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Presenting the Report of the Joint Commission to the National Council of Teachers of Mathematics

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The development of the mathematical curriculum of the secondary schools is divided into two periods, one from 1893 to 1923 and the other from 1923 to 1940. The first period begins with the report of the famous *Committee of Ten on Secondary Studies*, the second with the report of the equally famous *Committee on Mathematical Requirements*. Each report has made a lasting impression on the mathematical curriculum. The presentation of the report of the *Joint Commission* marks the beginning of a third period in the advancement of the course of secondary school mathematics. Its influence will soon be seen in the forthcoming textbooks and in the newer courses of study.

The commission has spent five years in intensive study of the problems of the teacher of secondary mathematics. The members were unusually well qualified for the task. Seven represent the *Mathematical Association of America*. They are University teachers. The other seven are either teaching in secondary schools or are closely related to the teaching of mathematics in the secondary schools. They represent the *National Council of Teachers of Mathematics*.

The fact that the Universities have taken an active interest in the work of the lower schools adds greatly to the value of this report.

From the beginning it has been the policy of the commission to draw the general teaching public into their discussions. Members of the commission have taken the problems to numerous local groups of teachers for criticisms and suggestions. Two years ago a tentative report was published. This was widely distribs uted with the request that the teacher- offer further suggestions and guidance. Finally, just a year ago, the National Council was asked to devote one section of the program at the Cleveland meeting entirely to further criticisms of the report. Thus, the commission has availed itself of every possible opportunity to study the trends in the teaching of mathematics in this country.

The commission was originally organized to take over the work of several very important committees which were trying to solve some of the problems relating to the teaching of mathematics. It has done more than that. It has gone back to the report of the National Committee of 1923 and has carried forward the work of that committee. In many respects the two re-

ports are similar. Thus to the brief chapter on aims in the report of 1923 corresponds a full discussion in three chapters of the *general* aims of education, of the objectives of *secondary* education, and of the place of mathematics in education.

Today when the question of the value of mathematics is raised so often by pupils parents, and administrators, the teacher needs to have a ready answer. He will find it in these three chapters. They are full of suggestions which will enable him to justify the teaching of his subject. Moreover, much of the ineffective and aimless teaching which may be observed is due to a lack of clearly defined objectives. An analysis of teachers' reports on the causes of failures discloses invariably the comment that the pupil is able to do the work but lacks appreciation and has a poor attitude toward the study. These objectives are not attained with many pupils unless the teacher makes a serious effort to reach them. The report of the commission not only emphasizes the importance of teaching for understandings of concepts and principles and for mathematical skills, but makes it clear that desirable attitudes and appreciations are no less essential.

The National Committee devoted three rather short chapters to discussions of the mathematical curriculum. It suggested four general plans, outlined in the brief space of less than two pages, and recommended further experimentation with these plans. The Commission also has three chapters dealing with this problem. However, it presents in detail two plans for the secondary period and gives two topical outlines for the junior college period.

Four other chapters in this report bring up to date similar chapters in the report of the National Committee. They are: (1) on "terms and symbols," found in Appendix III; (2) on "transfer of training," in Appendix II; (3) on "testing and evaluating progress in mathematical instruction," Chapter XIII; (4) on "Training of Mathematics Teachers," Chapter XIV.

Thus, the report of the commission carries forward the discussion of the major problems studied by the committee of 1923. It also presents three new problems: (1) on "Retardation and Acceleration of Pupils," in Chapter VII; (2) on "Mathematics needed in Ordinary Life, leadership and higher culture," in Appendix I; (3) on "Class room equipment, models, supplies, and mathematical library," Appendix IV.

Although the commission has gone beyond the work of the committee of 1923, it makes no claim of having solved or even attempted to solve all the problems in the teaching of mathematics. For example, it does not propose to recommend methods of teaching. It does not recommend a rigid program that all types of teachers or all types of schools should introduce. The proposed program is stated in terms of such broad categories as to be sufficiently flexible so that a large number of schools should find it possible to adjust their work to it. The categories are:

1. Number and computation
2. Geometric forms and space
3. Graphic representation
4. Elementary analysis
5. Logical thinking
6. Relational thinking
7. Symbolic representation and thinking

Each of these seven categories is subdivided into five smaller categories, as follows.

(1) Concepts, principles and terms
(2) Fundamental processes
(3) Fundamental relations
(4) Skills and techniques
(5) Appreciations

The curriculum provides continuity of instruction in many major ideas of mathematics. From the beginning mathematics is presented as a mode of thinking. Throughout the entire course emphasis is on such important objectives as habits and attitudes. Basic mathematical con-

cepts are introduced early and development is gradual until understanding is attained. The plan makes it possible to correlate the mathematical subjects with each other and with the related school subjects. Each year's emphasis is usually on one subject, but no subject is ever completely dropped. The pupils' mathematical experiences are thereby greatly enriched.

What will be the value of this report? The report will be carefully studied by educators and curriculum makers. It will be a guide to administrators and teachers of mathematics. However, the real value of the report depends on the extent to which it will be of use in the decade ahead of us. Hence, it must have first of all the backing of the members of the *National Council*. The officers of the Council have strongly endorsed the report by making it one of the year books. It will mean a great deal for the future of mathematics if the members likewise endorse it as individuals. It deserves more than to be merely read and then to be laid aside. It should be read frequently until the ideas and recommendations have been thoroughly assimilated. If this is done some much desired improvements in the teaching of mathematics will take place quickly. A single leader may advocate an idea for 25 years or more before it finds acceptance. A national report, nationally supported by the teachers, can bring this about in the time of months. In accepting the report, the National Council should remember that its success lies in the enthusiastic support of the individual members of the Council.

May Expirations

DURING the month of April, the majority of members of the national Council of Teachers of Mathematics will receive notices asking them to renew their subscriptions to THE MATHEMATICS TEACHER and memberships in the Council, which expire in May 1940.

If the October 1939 number of the TEACHER was the first on your current subscription, you will have received by May eight numbers, the complete issue for this school year, and, according to the Council's regulations, dues for the continuation of membership and for the TEACHER are payable upon expiration in May.

Please try to save the officers responsible for the Council and for THE MATHEMATICS TEACHER serious trouble by sending in your renewal promptly.

Three Major Difficulties in the Learning of Demonstrative Geometry

By Rolland R. Smith

Part II

DISCRIPTION AND EVALUATION OF METHODS USED TO REMEDY ERRORS*

Chapter V

PROCEDURE IN PART II

During the interval between the analysis of the results in Part I, and the beginning of the experiment for Part II of this study, methods of dealing with the three fundamental difficulties were devised and tried out on several successive classes. Elements of the methods which were awkward or inefficient were dropped and replaced by others. Finally, a complete course of study based on the new methods was written.

In the fall of 1938 a study was made to show that the new methods actually did what they were intended to do; that is, to help pupils with their difficulties. The usual method of procedure would be to set up two comparable groups, teach one group by the new methods and the other by the old methods, and then compare the groups. This was not possible here. All the mathematics teachers in the school had been working with enthusiasm trying out the new methods. None of them could have gone back to the spirit of the old methods. Besides, we did not consider it a fair proposition to teach any group of pupils by any but the best methods we knew. It was necessary, therefore, to make use of a different procedure.

The new experiment was set up in such a way that Group T of the 1932 study might be considered as the control group. From three classes 74 pupils (Group D) with the same mean I.Q., and same standard deviation as Group T, were chosen. This equated the groups so far as I.Q.'s are concerned. On each day of the 1938 experiment the teachers and pupils covered the same subject matter as on the corresponding day in 1932. By this is meant that day by day they took exercise for exercise, figure for figure, definition for definition, theorem for theorem, etc., in the new study as in the old. The only exceptions to this general rule are explicitly shown as the discussion proceeds. The members of Group D were younger, on the average, than the members of Group T, because geometry was now begun in the 10B (the first semester of the tenth year) class instead of the 10A (the second semester) class. The pupils of Group T had had a year and a half of algebra while Group D had had only one year. The method of dealing with the subject matter in 1938 was different from the methods used in 1932.

What evidence we have (see page 151) leads us to believe that Group D was certainly no better at the start than Group T. Since the I.Q.'s were equated, the same subject matter was used day by day, and, since age and mathematical experience were in favor of Group T instead of Group D, any superiority shown by Group D may be attributed to the methods.

* This is the second and final installment of Dr. Smith's thesis. The first appeared in the March issue of The Mathematics Teacher. Reprints of this thesis (65 pages) bound in cloth may be obtained postpaid for $1 from The Mathematics Teacher, 525 West 120th St., New York, N.Y.

It should be understood that the necessity for making valid comparisons between the results of Groups T and D made it impossible to follow the new course of study except for its methods. The new course is different from the old, not only in methods, but also in choice of exercises and the order of introduction of the subject matter. In many cases it wanders from the purely geometric in order to apply what has been learned to non-geometric situations. Because of the limitations of the experiment, the best that could be done was to superimpose the new methods on the old subject matter.

It should be understood too that we are not making a comparison of methods which attempts to get at meanings with traditional methods that disregarded meanings. The course of study used in 1932 was written in an attempt to get at meanings. The beginnings of the methods used in 1938 are to be found in the 1932 study. The comparison is between a good course, judged by the standards of the day, and a better course.

Tests 1, 2, and 3 were given to Group D, using the same methods as those used with Group T. In every case, Group D had poorer or equal results. We conclude, therefore, that Group D was certainly no better than Group T at the start.

The results of these three tests are shown in Table 26.

TABLE 26

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON TESTS 1, 2, AND 3

Test and Exercise	Percentage of Pupils Making Errors		Difference
	T	D	
Test 1 Bisecting lines			
1	0	0	0
2	6	9	−3
Test 2 Bisecting angles			
1	0	0	0
2	15	20	−5
Test 3 Constructing Perpendiculars			
1	0	1	−1
2	0	1	−1
3	1	3	−2
4	1	1	0
5	12	24	−12
6	11	22	−11
7	15	22	−7
8	8	9	−1
10	10	14	−4

In Test 3, Exs. 9, 11, and 12 were different in the case of Group D from those given to Group T (see Test 3, page 103, and Test 3a, page 153).

The same tests were given throughout to Group D as to Group T. However, only a few key tests have been chosen in order to make comparisons. These comparisons, together with a description of the new methods used, are given in succeeding chapters.

CHAPTER VI

ANALYZING FIGURES

IN CHAPTER II we gave evidence to show that one of the basic difficulties in the learning of demonstrative geometry is the inability of many pupils to transfer to a complex figure skills which they had acquired in connection with a simple figure. We found that slight changes in the figure affected the pupils' responses. Ability in connection with a simple figure did not insure ability in connection with a complex figure even though the complication was slight.

Methods devised to minimize the difficulties in connection with figures were based on the assumption that pupils at first see the figures as wholes. When they are required to make a construction in a complex figure, the lines and points to which they should give attention do not stand apart from the other points and lines. The application of a definition or a theorem to a complex figure is made confusing because the particular configuration, which should be dissociated from

the rest of the figure, is obscured by the picture as a whole.

To remedy this defect, analysis and generalization appear necessary. When pupils learn how to construct a line perpendicular to a horizontal line from a point not on the line, their attention is concentrated upon this particular figure, and the particular letters used to designate the point and the line. They do not necessarily analyze the essential procedure and then generalize it. They are more likely to think in particular, "With C as center, draw arcs on AB," than to think in general, "With the given point as center, draw arcs on the given line." Then, when they are confronted with the problem of constructing the altitudes of an obtuse triangle, they are confused by the larger number of points and lines, and the necessity of extending certain lines. They do not know which points to use as centers nor where to draw the arcs.

Similarly, when pupils learn to recognize alternate interior angles in a figure consisting of two parallel lines and a transversal, they cannot necessarily transfer this ability so that they can recognize alternate interior angles in a parallelogram with diagonals drawn. There are parts of the first figure that are not essential—for one thing the transversal not only meets the parallels, but goes beyond them in both directions. This is not true in the second figure. Such unessential details as this, coupled with the fact that the purely essential parts have not been sufficiently analyzed, cause confusion in the more complex figure.

In order to apply a theorem to a complex figure, pupils must see in that figure the particular configuration which comprises the essential relationships of the theorem. The natural tendency to see a figure as a whole conspires against this analysis. Teachers of geometry, therefore, are faced with the problem of devising methods which will help pupils to see the essential parts of a figure, to analyze them so that they will see them in a more complex figure, dissociating the relevant from the irrelevant. Any method that will foster the attitude of analysis and generalization should prove effective.

Constructions

The following two illustrations will indicate the method used in this study to lessen the difficulties in connection with constructions. The purpose of the method was to generalize the procedure of a construction so that it would apply to any figure rather than to a particular figure.

After pupils had learned how to bisect angle BAC in a particular figure, they were asked the following questions:

1. Where do you first put the point of the compass?
2. Where do you draw the first arc?
3. Where do you then put the point of the compass?
4. What do you do next?

The tendency at first was to answer in terms of the particular figure. The answers to the first two questions, for example, were: (1) on A, and (2) so that it intersects AB and AC. Pupils were then told that in order to make use of this analysis in bisecting angles in other figures, they should answer the questions in general terms without mentioning the particular letters. The following answers were then given.

1. On the vertex of the angle I am to bisect.
2. So that it intersects the sides of the angle I am bisecting.
3. In two places—at the points where the first arc crosses the sides of the angles.
4. I draw arcs that intersect within the angle.

Drawing the bisector from the vertex to the intersection of these two arcs was not mentioned in this generalization, since this step seemed to cause no difficulty.

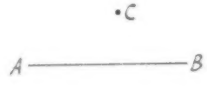
Pupils were then drilled upon the answers to these questions. Since they had previously been given practice in designating the vertices and sides of angles in

complex figures, this analysis and generalization of the procedure in bisecting an angle served as a bridge to allow transfer from the simple figure with which they had acquired the skill to more complex figures.

In connection with the construction of perpendiculars, even more generalization was possible. Pupils learned to construct perpendiculars in the following four cases, and then were shown that one general method applies to all of them.

1. At C construct a line perpendicular to AB.

2. From C construct a line perpendicular to AB.



3. At A construct a line perpendicular to AB.

4. From C construct a line perpendicular to AB.

In Exercises 3 and 4, the necessity of extending the line was discussed.

Then, as in the case of bisecting an angle, the procedure was analyzed and generalized by means of the following questions and answers:

1. In each case where do you first put the point of the compass? *Answer:* On the point mentioned.

2. Where do you draw the arc? *Answer:* On the line mentioned.

3. How many times must the arc cross the line? *Answer:* Twice.

4. If the arc does not cross the line twice, what do you do? *Answer:* Extend the line.

The rest of the construction was not analyzed, since it seemed to cause no difficulty.

In order to determine whether this method was effective, the following procedure was used:

For a home assignment on the seventh day, pupils in Group D were asked to study the method of constructing a line perpendicular to a given line at a point on the line and from a point not on the line. The textbook gave the method using figures like those in Exs. 1 and 2 above. In class the next day practice was given with these particular figures. Test 3a was then given. After the test the analysis and generalization indicated above was given and drilled on for five minutes only. Then the test was given again.

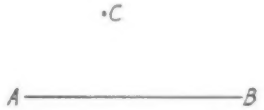
As a control group for this particular test, 74 pupils (the same number as in Group D) with the same range of I.Q.'s, the same mean and standard deviation, were chosen from three classes. (We shall call this Group E.) The same thing was done in Group E as in Group D, except that the pupils in Group E were given five minutes of practice on the four exercises (the first four in the test), instead of the analysis and generalization.

The results are shown in Table 27.

TEST 3a

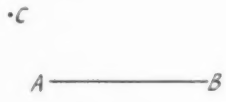
1. Construct a line perpendicular to AB at C.

2. Construct a line perpendicular to AB from C.

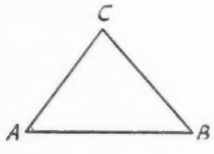


3. Construct a line perpendicular to AB at A.

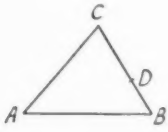
4. Construct a line perpendicular to AB from C.



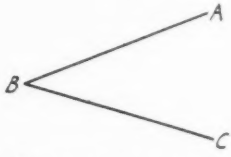
5. Construct a line from B perpendicular to AC.



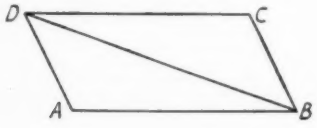
6. Construct a line from D perpendicular to AC.



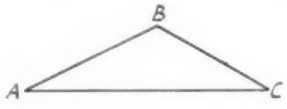
7. From B construct a line perpendicular to AB and another perpendicular to BC.



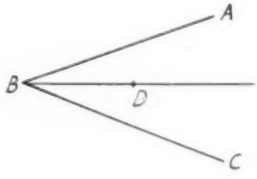
8. From C construct a line perpendicular to DB.



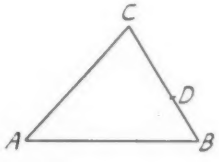
9. From A construct a line perpendicular to BC. From B construct a line perpendicular to AC. From C construct a line perpendicular to AB.



10. From D construct a line perpendicular to AB.



11. At D construct a line perpendicular to BC.



12. At D construct a line perpendicular to AD.

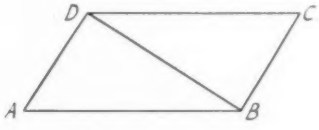


TABLE 27

COMPARISON OF THE IMPROVEMENT IN GROUPS D AND E IN TWO TRIALS OF TEST 3A

Group		Number of Pupils Making Errors on Exercise											
		1	2	3	4	5	6	7	8	9	10	11	12
D	First trial	1	1	2	1	18	16	16	7	25	10	10	13
	Second trial	0	0	0	0	5	4	3	1	11	3	5	3
	Difference (d_1)	1	1	2	1	13	12	13	6	14	7	5	10
E	First trial	0	1	3	2	19	15	13	9	28	10	10	13
	Second trial	0	0	2	1	13	11	5	5	19	8	7	7
	Difference (d_2)	0	1	1	1	6	4	8	4	9	2	3	6
d_1-d_2		1	0	1	0	7	8	5	2	5	5	2	4

Improvement is indicated in both groups. Greater improvement was shown in Group D in every exercise except Exs. 2 and 4, and in both of these exercises the pupils were nearly perfect in the first trial. The greater improvement in Group D is statistically significant as shown in the footnote.[1] Although the greater improvement in Group D is small it should be considered in light of the fact that the method of analysis and generalization had been used for a very short time only, five minutes.

One other procedure was tried in connection with the construction problems. The results of Test 4 given to Group T (see page 105) indicated lack of awareness that the order in which lines are drawn is of importance. For this reason, Test 4a was given to Group D, instead of Test 4, to discover the effect of more specific directions. Ex. 1 is the same on both tests. In Ex. 2 of Test 4a, pupils were told in what order to draw the lines. Comparison of the results is shown in Table 28.

TEST 4a

Ex. 2. Reproduce the figure below as follows: Draw a horizontal line *AB*, and place a point *G* on it. Construct an angle equal to angle 1, thus getting the line *EGF*. Place a point *H* on *EF* so that *GH* will equal *GH* in the given figure. Construct an angle equal to angle 2, thus getting the line *CD*.

[1] We may consider the twelve differences as twelve different observations of the same thing in testing the assumption that differences between the groups are due only to chance.

$$\Sigma(d_1 - d_2) = 40$$

$$M = 3.33$$

$$\Sigma(d_1 - d_2)^2 = 215$$

$$s^2 = \frac{215}{12} - (3.33)^2 = 6.84$$

$$\sigma_M = .789$$

$$t = \frac{M}{\sigma_M} = 4.22$$

For eleven degrees of freedom $t_{.01} = 3.106$ and the likelihood of exceeding $t = 3.8$ is .003. We see, therefore, that $t = 4.22$ may be considered significant.

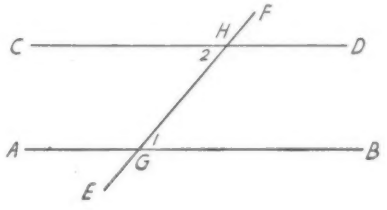


TABLE 28

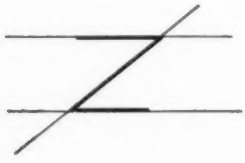
COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUP T MAKING ERRORS ON TEST 4 WITH THE PERCENTAGES OF PUPILS IN GROUP D MAKING ERRORS ON TEST 4A

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	4.4	2.7	1.7	2.8	.6
2	72.8	18.9	53.9	7.5	7.2

The difference in the results on Ex. 2 is striking. Difficulties can be avoided by giving careful directions as in Ex. 2 of Test 4a. To be sure pupils must finally be able to analyze a procedure sufficiently to do an exercise as stated in Ex. 2 of Test 4, but it is wise not to heap too many difficulties upon them at the start.

Application of Definitions and Theorems

Whenever a term is defined, it should be illustrated in complex figures as well as in a simple figure. When a proposition is proved, or otherwise accepted, pupils should have practice in seeing its application in complex figures. The teacher should find methods that will focus the attention of the pupils upon the particular parts of a figure which it is necessary for them to see. Colored crayons may be used to make applications stand out. Particular parts of figures may be drawn separately at the side of the complex figure. The entire figure can be drawn on the blackboard, and then the non-essential parts erased in order to show the essential parts. Some peculiarity of the situation such as the zigzag always formed by the alternate interior angles made by two parallels and a transversal may be mentioned explicitly. In every case, practice should be



given in seeing the application in a number of complex figures under the supervision of the teacher before there can be any great probability that all the pupils will be able to make the applications by themselves.

We shall record here the results of two tests given to show the effectiveness of some of the methods mentioned above.

On the thirty-first day Test 11 (see page 114) on the recognition of opposite angles formed by two intersecting straight lines was given to Group D. It was given originally to Group T immediately after the equality of the opposite angles had been shown in a figure consisting only of two intersecting lines. In the case of Group D, we went one step further. The teachers pointed out explicitly the two intersecting lines in the figure for Ex. 4, asked pupils to compare that part of the figure with the figure for Ex. 2, and to tell what angles were therefore equal.

Comparison of the results is shown in Table 29. The figures in which the opposite angles occurred were for Exs. 2, 4, 6, 9, 10, and 11. The differences are not statistically significant, except in one case.[2] However, they are consistently in one direction, and for this reason we may put some reliance in them. The small differences bear out the contention that practice should be given under the supervision of the teacher in the application of a theorem to several figures before pupils are asked to make the application on their own initiative.

[2] The likelihood of a difference of per cents exceeding two standard deviations by chance is about .05 and of exceeding two and a half standard deviations is about .01. Most authorities would not consider a difference significant unless it reaches at least two standard deviations, and many prefer two and a half standard deviations.

TABLE 29

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON TEST 12

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
2	0	0	0	0	—
4	15.8	5.4	10.4	4.8	2.2
6	14.0	6.8	7.2	4.7	1.5
9	10.5	4.1	6.4	4.0	1.6
10	16.7	6.8	9.9	5.0	2.0
11	19.3	4.1	15.2	5.0	3.0

On the forty-ninth day, Test 9 (see page 111), was given to the pupils of Group D. When the test was given to Group T, the members of this group knew the definition of alternate interior angles, but had had experience with them only in a figure consisting of two parallels and a transversal. In addition to this, Group D had been shown the zigzags by means of colored crayon, and had discussed Exs. 2 and 3 of the test, making use of colored crayon and noting the zigzags. Comparison of the results (see Table 30) shows that Group D was not so successful with Ex. 1, the figure with which the definition was given in both groups. But in spite of this handicap, Group D was unquestionably better on all the other exercises with the possible exception of Ex. 10.

TABLE 30

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 9

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	5.3	6.8	−1.5	3.5	−4.2
2	24.6	1.4	23.2	5.4	4.3
3	23.7	2.5	21.2	5.4	3.9
4	33.4	6.8	26.6	6.3	4.3
5	25.4	0	25.4	5.4	4.7
6	32.4	12.2	20.2	6.4	3.2
7	34.2	13.5	20.7	6.5	3.2
8	68.3	50.0	18.3	6.8	2.7
9	43.0	5.4	37.6	6.5	5.8
10	79.8	67.5	12.3	6.4	1.9

If we consider as correct those exercises where one pair of angles was given correctly, and the others not mentioned, the percentages of pupils in error would read as follows for Group D: Ex. 4 (3%), Ex. 6 (3%), Ex. 7 (6%), Ex. 8 (3%), and Ex. 10 (7%). Considering these percentages and comparing them with the corresponding percentages given on page 112 (they were 21, 26, 29, 15, and 25 respectively), we get even a better picture of the improvement. In the case of Group D, the errors of commission were found to be almost negligible in number.

Other specific methods of dealing with figures are suggested by Test 6, page 107; Test 7, page 109; Test 8, page 110; Test 10, page 113, and Test 12, page 116. Once a teacher is aware of the fact that pupils have difficulties with complex figures, the remedy is not hard to discover. In order to minimize the difficulties, he should make definite provision in his lesson plans for practice with figures, not only when a new topic is presented, but also later whenever it appears necessary.

Chapter VII

DEVELOPING THE MEANING OF THE IF-THEN RELATIONSHIP

From the results of the tests discussed in Chapter III we came to the conclusion that many pupils beginning demonstrative geometry have very little understanding of the logical implications of a proposition stated in the if-then form. For additional evidence toward the same conclusion, we gave the following test to Group D on the twelfth day.

Test 14a

What is the difference in meaning, if any, between the first two statements below?

1. If two sides of a triangle are equal, the angles opposite those sides are equal.
2. If two angles of a triangle are equal, the sides opposite those angles are equal.

Does either one of the statements above mean the same thing as the statement below? If so, which one?

3. In an isosceles triangle, two sides and two angles are equal.

Out of the seventy-four pupils in the group, sixty-one said there was no difference in meaning between the first two statements, six gave uninterpretable answers, and only seven gave correct answers. Fifty-five said that the third statement meant the same as "both the others," four said that it meant the same as 1 and 2 together, twelve the same as 1, and one the same as 2. Only two said that it was not the same as either. It is obvious that the group as a whole had very little conception of the difference in meaning between the first two statements. The realization of the difference in meaning involves an understanding of the if-then relationship.

It will be granted that pupils know what we mean when we say, "If you do excellent work in geometry, I will give you an A." But there is a difference between this and the geometric situation. This is dynamic. There is an element of time in it. The condition comes first in point of time. The conclusion comes afterward. There is also a causal element. Fulfilling the conditions in the if-part of the sentence in a sense causes the conclusion. The geometric situation is static. For either statement (1) If two sides of a triangle are equal, the angles opposite those sides are equal, or (2) If two angles of a triangle are equal, the sides opposite those angles are equal, the pupil sees the same figure in the textbook. What he sees is a triangle with two sides and two angles equal. So far as this static figure is concerned, there is no thought of the equal sides coming first and the equal angles coming afterwards, or vice versa. They are both there at the same time. Nor does it occur to him that the equal sides may be, in a

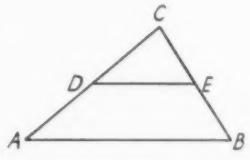
sense, the cause of the equal angles, or the equal angles the cause of the equal sides. For these reasons, the real meaning of the if-then relationship in a geometric proposition is hidden, and may not become clear in time for pupils to begin proofs with understanding. It should not be surprising to us that so many pupils flounder in the early stages of demonstrative geometry. When a pupil does not sense what it means, "If this is so, then that is so," the work he does must indeed be confusing and meaningless to him.

In order to develop the meaning of the if-then relationship, the author has introduced into the geometric situation the time and causal elements which are lacking. If a pupil constructs a figure according to certain conditions, he finds that other relationships result. Although both sets of relationships are true of the completed figure, they separate themselves into two categories. The pupil has control over one set. He made them so. He has no control over the other set. They came as a result of what he did. The given conditions were put into the figure first. The other relationships came afterward. Once a pupil sees that the relationships in a figure can be put into these two categories, the concept may be associated with the corresponding if-then statement. When the difference between the two categories is clear, the meaning of the if-then relationship has taken on concrete meaning.

The germ of this method was used with Group T (see page 120), but since its details had not been clearly worked out at that time, it was not used consistently, and had little effect. The method as used with Group D is described completely in the following paragraphs.

The development was begun on the thirteenth day with the discussion of the following exercises.

1. We have experiments in geometry just as we have experiments in science. We do certain things to a figure, and certain other things result. Here we shall bisect two sides of a triangle, connect the points of bisection, and see what will happen.



(a) Draw a line AB 8 cm. long.

(b) Complete triangle ABC with any dimensions you choose.

(c) Bisect AC and call the midpoint D.

(d) Bisect BC and call the midpoint E.

(e) Draw a straight line from D to E.

(f) Measure DE. How long is it? Do your classmates have the same result, even though the dimensions of their triangles are different?

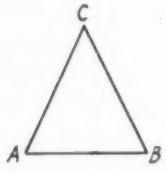
(g) Did you plan to make DE equal to 4 cm.?

(h) Did you know that DE is 4 cm. before you completed the figure, or did you learn this after you completed it?

(i) If you did the same work again accurately, with AB equal to 8 cm., could you make DE any length but 4 cm.?

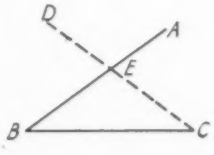
The important thing for you to learn from this experiment is that *if* you do certain things in making a figure, *then* certain other things inevitably result. Here you draw a triangle with a base 8 cm. long. You bisected the other two sides, and drew a straight line joining the midpoints of these sides. You had control over all these things; but you had no control over the *length* of the line connecting the midpoints. *Once you have done certain things you have no control over the result.*

2. Draw angle ACB, making CA equal to CB. Draw AB. You now have triangle ABC. Measure $\angle A$ and $\angle B$. What do you note about these angles? Did you plan to make $\angle A$ equal to $\angle B$? If you did this same work again accurately, could you bring about a result such that $\angle A$ would not be equal to $\angle B$?



In the figure you have just made $CA = CB$ and $\angle A = \angle B$. Which of these two equalities did you control? Over which did you have no control? We can say, then, that the equality of $\angle A$ and $\angle B$ is a result of the equality of CA and CB.

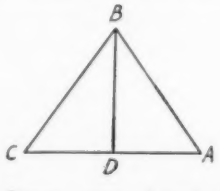
3. Draw an acute angle ABC. Then make an angle BCD equal to $\angle ABC$, as indicated by the dotted line. You now have a triangle BCE. Measure BE and CE.



In the figure you have just made $\angle B = \angle C$ and $BE = CE$. Which of these two equalities did you have under your control? In your figure would you say that the equality of $\angle B$ and $\angle C$ is a result of the equality of BE and CE, or vice versa?

4. Draw an angle ABC. Then, using B as center, mark off BA equal to BC. What kind of triangle is $\triangle ABC$? Bisect $\angle ABC$, and continue the bisector until it meets AC at D.

Measure AD and DC. Is AC bisected?



In the figure you have just made, $AB = BC$, BC bisects $\angle B$, and BD bisects AC. Which of these three things were under your control, and which was the result of what you did?

5. Draw $\angle ABC$ and then, using B as center, mark off BA equal to BC. Draw AC. Bisect AC, thus finding the midpoint D. Draw BD.

Measure $\angle ABD$ and $\angle CBD$. Is $\angle B$ bisected?

In the figure you have just made, $AB = BC$, BD bisects $\angle B$, and BD bisects AC. Which of these three things did you have under your control, and which was a result of what you did?

Note that the two figures you have made look very much alike, but that they were made differently. In the first of these two exercises, the *given conditions* (the part over which you had control) are $AB = BC$ and BD bisects $\angle B$. The *conclusion* is BD bisects AC. In the second exercise, the given conditions are $AB = BC$ and BD bisects AC. The conclusion is BD bisects $\angle B$.

The conditions and conclusion of an exercise may be stated briefly in an if-then sentence, the conditions being in the if-part. The conditions and conclusions of Exs. 4 and 5 may be stated thus:

(1) *If* $AB = BC$ and BD bisects $\angle B$, *then* BD bisects AC.

(2) *If* $AB = BC$ and BD bisects AC, *then* BD bisects $\angle B$.

The reader can see from the exercises themselves what the objective is. The first exercise introduces the distinction between given conditions and conclusion without mentioning the words. In the completed figure AB is 8 cm. (approximately, of course), and DE is 4 cm., but there can be a difference in attitude toward the lengths of the two lines. The pupil made AB 8 cm. long. He planned it that way. DE "turned out" to be 4 cm. He did not even know that it was 4 cm. until he measured it.

The next two exercises give practice with the same concept. By constructing the figures, pupils can see for themselves the equalities over which they have control, and distinguish them from the equalities over which they have no control, but which are a result of what they did. The use of converses such as have been em-

ployed here is effective in clarifying the distinction. The figures when completed look very much the same, but they were made differently.

The last two exercises carry the development to its conclusion. Now that the distinction between the parts of a figure over which the pupil has control, and parts which are a result of what he has done, has been made clear, the words *given conditions* and *conclusion* are associated with the two categories. Finally, the method of stating the distinction by an if-then sentence is introduced. Since the fundamental concept is clear, the method of stating it will be understood.

During the following days at least one construction exercise a day, exercises which had been discussed as constructions only with Group T, was treated according to the method described here. At the end of each discussion, pupils were asked to state the exercise as an if-then sentence.

When it came time to introduce the words *hypothesis* and *conclusion*, they were associated with the two categories already developed. The hypothesis refers to what is given, the parts of the figure over which the pupil would have control if he constructed it. Conclusion refers to what is to be proved, the parts of a figure over which he does not have control, the parts which result from what has already been done.

As a test of the efficiency of this method of developing the meaning of the if-then relationship, Test 17 (see page 121), was given to Group D. This was done on the thirty-eighth day under the same conditions as it was given to Group T, except that in the case of the latter group, the meaning of the if-then relationship had not been carefully developed. The test required the drawing of a figure and the writing of the hypothesis and conclusion in terms of that figure from three verbal statements. In neither group had such an exercise been required or discussed before. The superiority of Group D is shown in Table 31.

TABLE 31

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 18

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	58.6	10.8	47.8	7.3	6.6
2	36.8	16.2	20.6	6.8	3.1
3	42.1	17.6	24.6	7.0	3.5

All the differences in this table are statistically significant (see footnote to page 156). It appears certain that pupils who, without specific training for the kind of task required in these exercises, can draw a figure and write the hypothesis and conclusion in terms of a figure, understand the meaning of the if-then relationship. The drop in the number of errors is quite striking. Even among those who made errors there was not the confusion shown by Group T. For example, there were no irrelevant relationships brought in for Ex. 1, only two pupils made such additions for Ex. 2, and only three for Ex. 3.

CHAPTER VIII

DEVELOPING THE MEANING OF PROOF

DEVELOPING the meaning of proof is not a simple matter that can be done in a day or two. There are too many details for the pupils to assimilate. The evidence presented in Chapter IV led us to conclude that many pupils have no conception of what it means to draw a conclusion from a general statement, and an application of it. They wrote conclusions when the conditions were not fulfilled. They omitted conclusions when the conditions were fulfilled. Often the conclusions were en-

tirely foreign to anything in the given statements. Nor did they sense the restrictions necessary in a deductive science. In drawing conclusions concerning geometric figures, they did not hold to the data but assumed relationships from appearances. In their proofs they often used "manufactured reasons," statements that had not been accepted. It was clearly evident that they did not know "what it was all about."

It will be granted that these pupils had been making informal deductions for many years. But in all those deductions, they had probably never been so definitely restricted to verbal logic. They could bring to bear anything they chose—appearances, intuition, any of their senses, hearsay, prejudices, or what not. They made their deductions from the total situation or from any part of it that impressed them. Now for the first time, probably, they were required to attend to a definite part of the total situation. They must not assume anything concerning a figure merely from the appearance. They must hold themselves to the data. They must not think in terms of intuitive notions which had been growing up with them over a period of years. They must think only in terms of a restricted set of propositions. They had, therefore, not only to form new habits, but to inhibit old ones.

The methods used with Group D to develop the meaning of proof will be discussed under the headings Deduction, Acceptable Reasons, and Formal Proofs.

Deduction

The aim in the development of the meaning of deduction was to show the pupils the following: (1) a simple deduction involves three statements, a general statement, an application of it to some particular person or thing, and a conclusion; (2) no conclusion may be drawn concerning the specific person or thing unless there is an application, in other words, an exact fulfillment of the conditions, of the general statement; and (3) if the conditions are fulfilled there is always a conclusion. Besides developing the understanding of these facts, we wished to foster the habit of looking for the conditions which are to be fulfilled, to develop skill in detecting just what the conditions are so that they may know when they have been fulfilled, and finally to have them check each conclusion to see whether it is the conclusion called for by the reason they are using.

The development was begun on the fifteenth day with the giving of Test 18 (see page 123) on syllogisms. The results of this test are shown in Table 32.

TABLE 32

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 18

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
I	4.4	2.5	1.9	2.8	.7
II	.9	1.4	−.5	1.6	−.3
III	20.1	9.4	10.7	5.5	2.0
IV	25.2	24.3	.9	6.4	1.1
V	29.8	12.2	17.6	6.3	2.8
VI	70.0	50.0	20.0	7.2	2.8
VII	54.3	21.6	32.7	7.4	4.6
VIII	75.3	62.2	13.1	6.8	1.9

We learn two things from a study of this table in which only three out of the eight differences are statistically significant, and one of the differences is negative. First, that the development of the if-then relationship, which had been going on since the twelfth day, had some effect, but very little, on the pupils' ability to carry through these syllogisms correctly. And secondly, that at the beginning of the development of the meaning of proof Group D was not much, if at all, superior to Group T.

The development was continued on the sixteenth day. The method used will be shown by quoting the directions given to the teachers conducting the experiment.

"To the pupils. Suppose you know that any student who has room 323 for a home

room is a senior; that one morning you meet a new student, stop to talk to him, find out that his name is John Stuart, and in the course of the conversation learn that his home room is 323. What thought would come immediately to your mind? (Have the pupils write this statement so that you can see how many are right. They will probably all be right. Tell them that this is the way that the intelligent human mind works.)

"To the pupils. What you have done is to draw a *conclusion* from two statements. Written formally, it would look like this. (Then write the following on the board.)

1. If any student has home room 323, he (or she) is a senior.
2. John Stuart is a student and has home room 323.
3. Therefore, John Stuart is a senior.

"The third statement is called a *deduction* from the first two statements. We have *deduced* the third statement from the first two statements. We did not come to the conclusion by any means other than a thought process.

"Leave the first syllogism on the board, and write the following:

1. If any student has home room 323, he (or she) is a senior.
2. Mary Smith is a student and has home room 322.
3. Therefore, . . .

"Have the pupils write their conclusion. More than likely most of them will write, 'She is not a senior.' Discuss the matter with them. Show that there might easily be an overflow, and that some seniors may have room 322 as a home room. From what is stated, we do not know whether Mary is a senior or not. However, write the conclusion, 'She is not a senior' on the board, and put a cross beside it to show that it is wrong.

"There is no possible conclusion in this case. A deduction cannot be made. What I have written on the board is an example of wrong deductive reasoning.

"Leave the first two syllogisms on the board, and write the following:

1. If any student has home room 323, he (or she) is a senior.
2. Frank Allen has home room 323.
3. Therefore, . . .

"Have the pupils write their conclusion. Most will say, 'He is a senior.' Perhaps some one will say that Frank may be the teacher, and therefore there can be no conclusion. Write the conclusion, 'He is a senior' on the board, and put a cross beside it.

"Leave these syllogisms on the board, and write the following:

1. Any child in this town under twelve years of age found on a public street after 9:00 P.M. is immediately sent home.
2. John Smith, a child eleven years old, was found on June Street at ten o'clock.
3. Therefore, John Smith was immediately sent home.

"Ask. Is this an example of good deductive thinking? Point out that it is not. Probably many of your pupils will do this for you. We do not know whether it was A.M. or P.M., and we do not know whether June Street is a public street. Put a cross against it.

"We have seen, from studying the situation involved in each of these exercises, which of our conclusions were correct and which were not. Now we should analyze each to see if we can tell, by giving attention to the statements alone, when we can draw a conclusion correctly and when we cannot. Could we have told by careful reading of the statements that we could make a deduction in the first exercise and could not in the remaining exercises?

"Discussion of the first syllogism. Have the pupils note that the second statement *fits* the if-part of the first statement exactly. Note that the third statement *fits* the then-part of the first statement exactly. When this occurs, we have an example of good deductive reasoning.

"What are the *conditions* of the first statement? (*Answer:* The person must be a student, and he must have home room 323.) There are two conditions.

"Are these two conditions fulfilled?

(*Answer:* Yes. The second statement says that John Stuart is a student and that he has home room 323.)

"What is the conclusion of the first sentence? (*Answer:* He is a senior.)

"Is this carried out exactly in the third statement? (*Answer:* Yes. It says that John Stuart is a senior.)

"Summary. The second statement must fit the if-part of the first statement exactly (it must fulfill its conditions). The third statement must fit the then-part of the first statement (that is, it must carry out the conclusion exactly as it is stated).

"Now go on to the second syllogism. Does the second statement fit the if-part of the first statement. (*Answer:* No.) Why not? How many conditions are there to be fulfilled? (Two. See above.) Are they both fulfilled? Which one is fulfilled, and which one is not fulfilled? Can we make a deduction in this case?

"Treat the third syllogism in the same way, then repeat the summary.

"Treat the fourth syllogism in the same way, except to show that the conditions are found in the subject, and all its modifiers, of the first sentence and the conclusion in the predicate.

"Emphasize the summary."

As already stated, it is the tendency of pupils, prior to their formal work in geometry, to make their deductions, using the total situation or whatever part of it impresses them. The preceding syllogisms were chosen to make use of this method of reasoning. It was possible to show pupils that they were wrong in their conclusions, particularly in the second and third exercises, by discussing with them the concrete situation involved. No careful analysis of the statements themselves was necessary to show the pupils their errors. The analysis of the statements, the concept of conditions and their fulfillment, the realization that the statements themselves held the key to the correctness or incorrectness of the conclusions, the introduction of the terms to be used came afterward. Pupils could understand these abstract things because they had first seen the incorrectness of their conclusions concretely.

On the seventeenth day, the following summary was given:

1. A deduction consists of three statements, (a) a general statement, (b) a specific statement which is an application of the general statement, and (c) a conclusion.

2. No conclusion can be drawn from the first two statements unless the second statement fulfills exactly the conditions of the first statement; that is, *fits* exactly the if-part of the first statement.

3. If the second statement fulfills the conditions of the first statement, then there always is a conclusion and this conclusion must follow exactly the then-part of the first sentence.

Then the test on syllogisms (Test 18) was discussed. In each case the general statement was changed to the if-then form, and pupils were asked to consider each exercise in the light of the summary given above. In the cases where no conclusion could be drawn, pupils were asked to change the second statement so that a conclusion could be drawn. Then they made the necessary conclusion.

Pupils were cautioned that they must not use their imaginations in making these deductions. They must not bring in extraneous matter. They must concentrate on the given sentences and restrict their thinking to what is explicitly stated.

On the eighteenth day, the following exercises were discussed as applications of the preceding development.

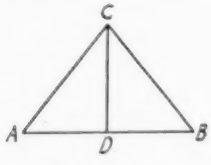
Study the examples given here and tell whether they illustrate right or wrong deductive reasoning.

I. 1. If two straight lines intersect, the opposite angles are equal.

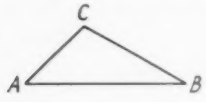
2. AB and CD in the figure below are straight lines intersecting at O.

3. Therefore, $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$.

II. 1. If a line bisects the vertex angle of an isosceles triangle, it is perpendicular to the base.
2. In the figure below, $AC = BC$ and CD bisects $\angle C$.
3. Therefore, CD bisects AB.



III. 1. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
2. $AB = DE$, $\angle B = \angle E$, $BC = EF$.
3. Therefore, $\triangle ABC \cong \triangle DEF$.



IV. 1. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent.
2. $AB = DE$, $AC = DF$, and $\angle C = \angle F$.
3. Therefore, $\triangle ABC \cong \triangle DEF$.

The postulates concerning congruence had previously been accepted after inductive treatment (see page 166).

Teachers were asked to comment on Ex. IV above as follows: It looks at first glance as if the conditions were fulfilled. We have two sides and an angle of one triangle equal to two sides and an angle of the other. But when we look at the figure we see that the angle is not included between the sides. One of the conditions is not fulfilled, and so there can be no conclusion. It should be clear that the only reason that we look at the figure is to see the relative positions of the lines, points, angles, etc.—to see where they are. We do not look at the figure to see whether the lines or angles are equal.

In order to test the effectiveness of the development to this point, Test 19 (see page 126) was given on the nineteenth day. In each of the two exercises the first statement was false, but its conditions were explicitly fulfilled, and so a conclusion was necessary. No such exercise had been previously discussed. In order to make the correct responses, a pupil would have to inhibit his natural tendency to reason intuitively, make use of what he had learned about deduction, and follow the words of the given statements without recourse to other considerations. The results are shown in Table 33.

TABLE 33

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 20

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	5.3	4.1	1.2	3.2	.4
2	55.2	39.2	16.0	7.5	2.1

The number of pupils in either group making errors on the first exercise is so small that comparison is rather meaningless. The difference in the second exercise is on the border line of statistical significance. It is probably safe to say that the recorded difference is not entirely due to chance, and yet at the same time we get an insight from this table as to how difficult the abstract notions of deduction are for pupils beginning demonstrative geometry.

The test was discussed as follows. If the

conditions of the general statement are fulfilled, there *must be* a conclusion and the conclusion must be the conclusion called for by the then-part of the sentence. Of course, the first statements of these two exercises are foolish. Yet, at the same time, their conditions were fulfilled. What we are trying to do is to show you what it means to make a deduction. We gave these foolish statements in order to emphasize the method, and to help you to remember it better. Assuming that the first statements are true (and that is what you are asked to do), there must be a conclusion to these two exercises.

Acceptable Reasons

Early in the course in demonstrative geometry, the pupil is confronted with the question, "Why?" He may be asked, "Why are these triangles congruent?" The teacher knows the meaning of *why* in this case; the pupil usually does not know it. He may have been asked a short time before, "Why are you unprepared?" and he told *how it came about* that he did not do his home lesson. He is to be pardoned, then, if he interprets the teacher's question to mean, "How did it happen that these triangles are congruent?" An appropriate answer would be, "The artist who made the drawings for the book was instructed to make them congruent." Then again, he may interpret the question to mean, "For what purpose are these triangles congruent?" The answer to this question he must confess he does not know. A better question would be, "What do you know about these triangles to make you sure that they are congruent?" Even then, the pupil may not give the correct response.

The difficulty lies deeper than the form of the question. The teacher means, "What postulate, theorem, or definition, previously accepted in this course, the conditions of which have been fulfilled, can you cite as an authority for the conclusion you have just reached?" This question would be explicit, but the meaning would not be clear to the pupil. He has made many informal inferences during his life, but he has not been restricted as to his premises. Now, perhaps for the first time, he is restricted to a very narrow set of acceptable reasons. He should be made aware of this limitation before he is required to make formal proofs in geometry.

Pupils can be shown that not all evidence is allowed in court proceedings. For example, a lie detector may be used prior to a trial in the hope that indirectly some evidence admissible in court may be found, but the direct evidence may not be admitted. If a lawyer should attempt to bring in such direct evidence, the judge would say in effect, "Your evidence (reasons) may be convincing and seem reasonable to you, but according to the law of the land I may not listen to you." It is, therefore, ruled out of court. This is only one example of the limitations placed upon lawyers in pleading their cases. There are many other examples. The lawyer must play the game according to the rules.

After a brief introduction like that just given, the pupil is prepared to hear that in the arguments he will henceforth give in geometry there are certain rules of good deductive thinking he must follow. One of them concerns the restriction in reasons he may use. He is ready for a discussion as to what constitutes an acceptable reason.

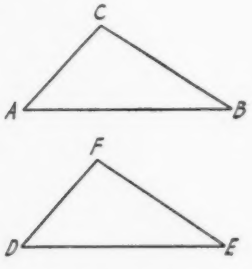
At the beginning of deduction in geometry, the only acceptable reasons (in addition to the particular hypothesis and construction of the given exercise) are definitions, postulates, and axioms.[1] Inferences from definitions should have been part of the pupil's work from the very beginning of his geometry course in connection with constructions and study of

[1] The writer makes the distinction between *axioms* and *postulates* that is current in secondary school teaching. An axiom is an assumption concerning quantities in general. A postulate is an assumption in the particular field of geometry. There is some psychological advantage in the distinction, but it is not necessary.

figures; hence at this stage it remains only to state explicitly that a definition is at all times acceptable in a deductive argument.

The following illustration will indicate how the postulates were introduced with Group D. First, the definition of congruent triangles was given as "triangles which have three sides of one equal to three sides of the other, and three angles of one equal to three angles of the other." Then followed construction work and discussion as shown below.

Draw any triangle ABC. Construct a line DE equal to AB, then construct $\angle D$ equal to $\angle A$ and DF equal to AC (see the figure below). Draw EF. Measure BC and EF, $\angle C$ and $\angle F$, $\angle B$ and $\angle D$. You see that three sides and three angles of one triangle are equal respectively to three sides and three angles of the other. Triangle DEF, according to the definition of *congruent*, is therefore congruent to triangle ABC.



1. How many sides of one triangle did you construct equal to sides of the other? How many angles of one did you construct equal to angles of the other?

Note that although you made only two sides and one angle of one triangle equal two sides and one angle of the other, the result was two triangles with three sides and three angles of one equal to three sides and three angles of the other.

The angle must be in a definite position with respect to the two sides—it must be the angle formed by the two sides. The angle made by two sides of a triangle is said to be *included* between these two sides.

2. What conditions were fulfilled to make these two triangles congruent? (Two sides and the included angle of one were made equal respectively to ....) Write as an if-then sentence the relationship illustrated in this construction exercise.

This was done on the seventeenth day.

Similarly, other geometric "facts" were introduced until we had the following list. Pupils wrote this list under the heading *postulates* in their notebooks.

1. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent.

2. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, they are congruent.

3. If two triangles have three sides of one equal to three sides of the other, they are congruent.

4. Any quantity is equal to itself (identity).

5. If one straight line meets another, so as to form adjacent angles, these angles are supplementary.

6. If two adjacent angles form a right angle, they are complementary.

7. If two angles are supplementary to the same angle or to equal angles, they are equal.

8. If two angles are complementary to the same angle or to equal angles, they are equal.

9. If two straight lines intersect, the opposite angles are equal.

10. All right angles and all straight angles are equal.

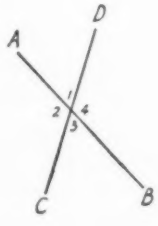
Axioms were also introduced by inductive methods.

As theorems were proved, another list was begun and added to. Pupils always had before them these lists of propositions which were "acceptable reasons." If they used others, they knew they were going against the rules of deduction, and would be "called for a foul."

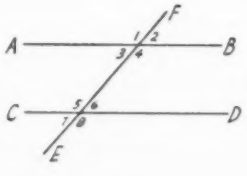
In order to show pupils how these

axioms and postulates are used, to prepare them for the later use of "Why?", questions were first given in a form which would tend to elicit the correct response. The following are illustrations:

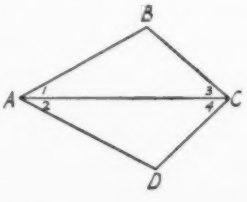
1. Since $\angle 2$ and $\angle 4$ are both supplementary to $\angle 1$ in this figure, they are equal. Which of the statements in your list would you give as an authority for this equality. (The list contained only items 1–8 at this point.)



2. In the figure AB and CD are straight lines intersected by line EF. Suppose you are told that $\angle 5$ is supplementary to $\angle 3$, and are asked to show that $\angle 5 = \angle 1$. Are $\angle 5$ and $\angle 1$ supplementary to the same angle? What is your authority for saying that $\angle 5 = \angle 1$?



3. If you were told that $\angle 1$ and $\angle 2$ are each 30° and AB and AD are each 2 cm., what could you find out about this figure



without further measurements? You know that $AB = AD$ and $\angle 1 = \angle 2$. You see that AC is the same line in both triangles. How many sides of one triangle do you know to be equal to sides of the other? How many angles? Is the angle included between the two sides? Have you sufficient information now to know that the triangles are congruent? What authority in your list would you give as a reason that they are congruent? If the triangles are congruent, what sides and angles do you know to be equal in addition to those you knew about at first?

4. Construct an isosceles triangle and bisect the vertex angle. It appears that the base is also bisected. Explain by means of congruent triangles why this has to be true.

During the time of this development, pupils were constantly reminded that the purpose of our work was to see how these new things grew out of the postulates. They readily accepted the restrictions as to reasons. When some pupil gave a manufactured reason as an authority, he was at once called to account by his classmates. Because of the kind of development given, he was convinced of his error as soon as it was called to his attention.

As a check on the efficiency of this method of showing that reasons must be restricted to those which have been accepted, Test 22 (see page 129) was given to Group D. The fourth exercise is the one in which we are interested here. When the test was given to Group T, 48% of the pupils gave unacceptable reasons. Only 20% of Group D gave such reasons. Fourteen per cent of Group D gave no reason rather than give one not on the list when they could not seem to decide upon the correct reason. Sixty per cent of Group D were entirely correct on this exercise as compared to 49% of Group T.

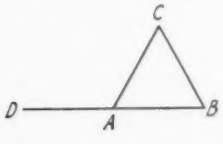
Formal Proofs

All the development so far described was made use of in the discussion of formal proofs. Figures were studied in connection with the postulates, and later with the theorems. Knowledge of the meaning of the if-then relationship was used in connection with the writing of the

hypothesis and conclusion from a verbal statement, and in understanding the implications of the reasons given. The development of the meaning of deduction was used as a basis of checking proofs to see whether they were correct. And, of course, pupils were allowed to give only acceptable reasons.

The first formal proof was introduced on the twentieth day by means of the following exercise. The bulletin given to the teachers for this day is reproduced to show the method in detail.

"Suppose we are given this figure in which we are told that $AC = AD$ and $BC = AD$ and are asked to show without measuring that triangle ABC is isosceles. That is, we are asked to prove that if. . . . (Have pupils give this if-then statement in terms of the figure.)



"Discuss the proof informally, holding the pupils accountable for acceptable reasons.

"The proof written out in form would look like this.

Hypothesis. $AC = AD$, $BC = AD$.
Conclusion. $\triangle ABC$ is isosceles.
Proof.

Statements	Reasons
1. $AC = AD$, $BC = AD$.	1. Hyp.
2. $AC = BC$.	2. If two things are equal to the same thing, they are equal to each other.
3. $\triangle ABC$ is isoceles.	3. If a triangle has two equal sides, it is isosceles.

"Now let us analyze this proof. The hypothesis refers to what is given concerning the figure, the part that is put into it, the part over which we would have control if we constructed it, the if-part of the sentence.

"The conclusion refers to the result, the part over which you have no control, the things which happen because of the hypothesis, the things that happen because of the things which were put into the figure. The conclusion contains what you are to prove.

"We know that we are restricted as to the reasons we may use. For the present we can use only the hypothesis, any definition we know, the axioms and postulates in our list. This is one of the most important rules in the game. If we use any others, we shall be called for a foul. From time to time we shall add to the list of reasons we may use.

"Drill. What are the three things we can use for reasons?

"Now look at the reason column. Have we used any 'unacceptable' reasons? What is the first reason? (Hyp.) What is the second reason? (An axiom.) What is the third reason? (A definition. It tells what an isosceles triangle is.)

"The first step in checking a proof is to see whether all the reasons are acceptable.

"The second step in checking a proof is to see whether the conditions of each reason have been fulfilled. (Do this now.)

"The first reason is 'Hyp.' This is the reason for statement 1. Is statement 1 found in the hypothesis? Is it actually given as you say it is?

"What are the conditions of the second reason? (We must have two things equal to the same thing.) Do we have two things equal to the same thing, and have we written that we have them? Yes, in statement 1.

"What are the conditions of the third reason? (We must have a triangle with two equal sides.) Do we have two equal sides, and have we written that we have them? (Yes, in statement 2.)

"The third step in checking a proof is to see whether we have followed exactly

the conclusion, the then-part, of each reason.

"What is the then-part of reason 2? (They are equal to each other.) What does 'they' refer to? Does the statement opposite this reason; that is, statement 2, follow this then-part exactly?

"Do the same thing with reason 3.

"Write the proof in syllogistic form as follows, to show pupils explicitly that the reasoning is exactly the same as that with which they dealt in the development of deduction.

I. 1. If two things are equal to the same thing, they are equal to each other.
   2. $AC=AD$, $BC=AD$ (Hyp.)
   3. Therefore, $AC=BC$.

II. 1. If a triangle has two equal sides, it is isosceles.
   2. $AC=BC$ (just proved).
   3. Therefore $\triangle ABC$ is isosceles."

From this point on there was nothing fundamentally new in the development of the meaning of proof. New subject matter was introduced, new postulates accepted, theorems were proved and listed as acceptable reasons. The method of analysis was carefully developed, but since it was discussed in the same way with Group D as with Group T, it does not affect the comparisons we are making, and so will not be reported here.

At first all exercises were simple. Many of them were discussed according to the method just described. It was the aim to show that each proof could be divided into separate syllogisms and each one checked to see if it was good reasoning. At least once a day for several days a proof was checked by the use of the following three questions.

1. Are all the reasons acceptable ones?
2. Are all the conditions of each reason fulfilled? (In case the reason was "Hyp." this question was interpreted to mean, "Is the statement actually in the hypothesis?")
3. Is the conclusion of each reason followed exactly?

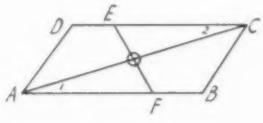
The way in which the check works can be seen by using it on the following "proof" taken from a pupil's paper.

Hyp. $AB=DC$, $AD=BC$, EF is a straight line passing through O, the middle point of AC.

Con. $FO=OE$.

Proof.

Statements	Reasons
1. $AB=DC$, $AD=BC$	1. Hyp.
2. $AC=AC$	2. Identity.
3. $\triangle ABC\cong\triangle ADC$.	3. s.s.s. = s.s.s.
4. $\angle 1=\angle 2$.	4. Equal angles formed by a diagonal.
5. $AO=CO$.	5. Hyp.
6. $\triangle AOF\cong\triangle EOC$.	6. a.s.a. = a.s.a.
7. $FO=OE$.	7. C.p.c.t.e.



Are all the reasons acceptable ones? A glance at the list shows that the reason for statement 4 is not acceptable. This is a danger signal. Perhaps the statement is correct, and an acceptable reason can be found. Perhaps the statement is wrong, and another method must be sought. In this case, we can find the proper reason. *Corresponding parts of congruent triangles are equal.* All the other reasons are acceptable.

Are the conditions of each reason fulfilled? We find no trouble until we get to reason 6. How many conditions are there? Four. (1) One angle of one triangle must be equal to one angle of the other, (2) another angle of one triangle must be equal to another angle of the other, (3) a side of one triangle must be equal to a side of the other triangle, and (4) the side in

each case must be included between the two angles. A check shows that we are missing one angle. Can we find it?

Is the conclusion of each reason followed exactly? Yes. For example, identity means, "Any quantity is equal to itself," and we have stated $AC = AC$. The conclusion of reason 6 is, "The triangles are congruent," and we have stated "$\triangle AOF \cong \triangle EOC$."

Pupils should realize that this method of checking does not tell them *what* is right. It merely shows them *when* something is wrong. They can know by this method of procedure, when they are right and when they are wrong. The danger signal should lead them to further analysis. They should realize that in case they cannot find a proof, it is better to leave a blank page than to fill it with errors. The first would show merely that they were not clever enough to find a proof; the second that they do not know what it means to make a demonstration.

We wish to reiterate that throughout the work with Group D, the same subject matter was discussed day by day as with Group T. The same textbook was used, the same exercises were discussed in class, the same exercises were given for home assignment. What was different was the method of approach. Whatever evidence we have shows that Group D was certainly no better mathematically than Group T at the start. The superiority of Group T, then, in later work we attribute to the difference in method.

By the time that the final test on originals was given (Test 25, see page 133), Group D was doing work superior in every way to that of Group T. This was shown not only in their written work, but even more so in the discussion of their proofs. We have no way of showing here the improvement in the discussions for we had no tests in which Group T did this kind of thing. We can, however, show the superiority of Group D over Group T in the results of Test 25, which are given in the following table (Table 34).

TABLE 34

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 25

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	54.4	17.6	36.8	7.3	5.0
2	42.9	17.6	25.3	7.0	3.6
3	68.4	39.3	29.1	7.4	3.9
4	45.6	31.0	14.6	7.3	2.5

The tests in this study were constructed on the assumption that geometry is taught in the senior high school to show a pupil what a demonstration is, and to give him a notion of what is meant by postulational thinking. The results of the tests have shown a multiplicity of errors, but we have been able to group most of these under a few main headings. The remedial methods presented here do not assure a pupil's ability to prove a difficult original exercise, but do give him a method of checking a proof once it is made either by himself or by another.

To understand a proof, to be able to check its validity is one thing. To be able to make a proof when the situation involves new factors is another. Obviously ability to do the one must precede ability to do the other, and the clearer the understanding of the former, the greater the chance of success in the latter. In some cases, pupils may never get far beyond the earlier stage which is one of appreciation, but they have then certainly fulfilled one of the fundamental aims in teaching geometry. To be able to understand what a demonstration is, and whether a proof is correct, should be of value to large numbers of pupils who would not profit by an extended course in geometry where the aim is to develop ability to do more or less difficult originals. Many pupils, who by traditional methods have been unable even the reach the appreciation stage, may be helped by the use of the careful development of meanings outlined in this study.

Chapter IX

TRANSFER OF TRAINING

In this study there have been many instances of transfer of training as well as the lack of it. The purpose of this chapter is to place the study against a background of the theories of transfer and to discuss the reactions of the pupils in terms of those theories.

Theories of Transfer

Geometry was taught for centuries as a mental discipline. "Just as we have legs for walking, eyes for seeing, and a tongue for talking—different parts of the body for different bodily functions—so it was supposed there might be 'parts' in the soul with functions or duties of their own."[1] The mind was supposed to have separate parts for such functions as sensation, memory, imagination, will power, and reasoning. These faculties could be trained just as muscles are hardened by suitable practice and when trained could be used without loss of efficiency in any field of endeavor. Geometry was the subject, par excellence, for training the reasoning power.

Faculty psychology goes back to the time of Plato and even before. The location of different mental functions in different parts of the body was attempted by several of the Pythagorean philosophers. The scholastics of the Middle Ages gave it shape and made a dogma of it. It remained supreme until the beginning of the present century even though Locke and Herbart made some protests against it in the eighteenth century.

In this country the attack against faculty psychology began with James' arguments against phrenology. It was he also who carried on the first experimental work to determine whether transfer is a reality. The death blow was given by Thorndike and Woodworth in 1901. Their conclusion was, "Improvement in any single mental function rarely brings about equal improvement in any other function, no matter how similar, for the working of every mental function-group is conditioned by the nature of the data in each particular case."[2]

Since this experiment by Thorndike and Woodworth, many others of similar nature have been made.[3] In each of these experiments the conclusions were much the same. Although in most of them some transfer took place, they "clearly demonstrated that, whether the processes were simple or complex, the effects of special training were transferred to a far more limited degree than previous educationists had supposed."[4] For this reason, among others, the theory of mental faculties and formal discipline was abandoned.

It could not be denied, however, that in most of the experiments transfer from one mental task to another was found in a measurable degree. Consequently, the problem of later investigators has been, not whether transfer takes place but under what circumstances it occurs.

It was early seen that when the amount of transfer was great, there was a close similarity between the two tasks and that when the similarity decreased the amount of transfer also decreased. This observa-

[1] Burt, Cyril, "Historical Note on Faculty Psychology," *Secondary Education with Special Reference to Grammar Schools and Technical High Schools*, Report of the Consultative Committee on Secondary Education with Special Reference to Grammar Schools and Technical High Schools (London: His Majesty's Stationery Office, 1939), p. 429.

[2] Thorndike, E. L. and Woodworth, R. S., "Influence of Improvement in One Mental Function upon the Efficiency in Other Functions," *Psychological Review*, VIII (1901), 250.

[3] Summarized by Blair, Vevia, "The Present Status of 'Disciplinary Values' in Education," *The Reorganization of Mathematics in Secondary Education*, A Report by the National Committee on Mathematical Requirements (The Mathematical Association of America, Inc., 1923), pp. 89–96.

[4] Burt, Cyril, *Op. Cit.*, p. 434.

tion led to the theory of identical elements as summarized in the following quotation. "A change in one mental function alters any other only so far as the two functions have as factors identical elements. The change in the second function is in amount that due to the change in the elements common to it and the first."[5]

This early theory of transfer had as its background "association" psychology which emphasized the one-to-one relation between specific stimuli and specific response. It explained transfer in a purely mechanical fashion. When there were identical elements in the external situation there was the probability of transfer, otherwise not. And yet it has been established that such things as ideals, attitudes, and generalizations transfer to situations where the identical elements are not apparent. It appears, therefore, that the identical element theory explains transfer in only a limited number of cases. The experiments of Lashley point toward a lesser emphasis upon the one-to-one relation between stimuli and responses as an explanation of mental reactions.[6]

Later investigators have liberalized this theory to allow for identical elements which are subjective as well as objective. The common element may be a method or an ideal which has been freed from a particular situation and so usable by conscious effort on subject matter of a different kind. It is generally agreed that transfer may be most effectively assured by methods of teaching which make the methods or ideals learned during the training period clearly conscious and freed of their context. Says Burt, "The common elements may be elements of (1) material, (2) method, (3) ideal. A common element is more likely to be usable if the learner becomes clearly conscious of its nature and its general applicability."[7]

[5] Thorndike, E. L., *Educational Psychology* (New York: Teachers College, Columbia University, 1903), p. 90.

[6] Lashley, Karl S., *Brain Mechanisms and Intelligence* (Chicago: University of Chicago Press, 1929).

Of nearly the same import is the following conclusion of Jordan made after analyzing several experiments on transfer. "It seems fair to conclude, therefore, that the development of meaning and understanding in material learned brings about a condition very conducive to transfer to functions similar in material and procedure."[8]

In opposition to the identical elements theory, Judd has taken the view that the most important instances of transfer are due to what he calls *generalized experience*. He has championed the view that particular ideas and methods may by suitable teaching be generalized so that they are no longer associated only with the material with which they are obtained. The method of teaching is all important according to this theory. "The type of training which pupils receive is determined by the method of presentation and by the degree to which self-activity is induced rather than by the content." Also, "Any subject which emphasizes particular items of knowledge and does not stimulate generalizations is educationally barren." And again, "It is the nature of generalizations and abstractions that they extend beyond the particular experience in which they originate."[9]

Stress is given to the development of meaning and understanding and the freeing of concepts from the context in which they are developed. In this connection, Bode says, "Human beings have the capacity of detaching the thing thus suggested or indicated (meaning) and treating it independently, apart from the specific setting." And a little later, "The great advantage in detaching meanings in this way is that they become more readily available in a variety of situations."[10]

[7] Burt, Cyril, *Formal Training*, Report of a committee appointed by the British Association and presented at Bristol, 1930. P. 4.

[8] Jordan, A. M., *Educational Psychology* (New York: Henry Holt and Co., 1933), p. 225.

[9] Judd, C. H., *Psychology of Secondary Education* (Boston: Ginn and Co., 1928).

[10] Bode, Boyd H., *Fundamentals of Education* (New York: The Macmillan Co., 1921), p. 152.

The most recent explanation of transfer of training is that given by the investigators in gestalt psychology. They conclude that the most significant things in our environment are not elements but the relations between these elements. There can be generalities in the forms of awareness, tendencies, attitudes, mental sets, and dispositions based upon relationships and not upon the specific nature of any particular situation or element in it. These can be considered general since they can be obtained from any one of many situations which are similar only in the relations involved and can be transferred to other situations which contain these same relationships. What transfers is a *pattern of experience.*[11]

All these theories of transfer may be confusing to the teacher of geometry. Yet there is a core of sameness in them that points toward a workable basis for teaching. Of one thing we can be sure. No great amount of transfer is automatic. If we wish the teaching of geometry to result in anything more than the learning of specific things in geometry, we must know what we are striving for and then search for suitable methods of teaching. The fact that there are identical elements in geometric reasoning and deductive reasoning elsewhere will not necessarily result in automatic transfer from the one situation to the others. The pupil may not note the similarity. He may not be sufficiently conscious of the method of reasoning in geometry to see the similarity. The relationships involved may not be seen clearly and they may not be freed from their context. The pupil may experience geometric reasoning but not reasoning in general. If we hope for transfer we must teach for it. Meanings must be developed. The methods of reasoning must be brought to the level of awareness. The logical relations developed should be made available in non-geometric situations by using them in a variety of contexts.

[11] Commins, W. D., *Principles of Educational Psychology* (New York: The Ronald Press, 1937), p. 432.

Application of the Theories of Transfer to this Study

The transfer shown in connection with constructions and the recognition of particular parts of geometric figures may well be explained by the presence of identical elements. After learning how to bisect a horizontal straight line, only 6% of Group T and 9% of Group D (see Table 26, page 151) were unable to bisect the sides of a given triangle correctly. The two tasks are obviously very much alike in detail and the transfer was large. It is important to note, however, that there were some who could not perform the second task in spite of the great similarity to the first. The identical elements did not insure transfer.

As shown in the results of further tests (see particularly Test 3, page 103 and Table 3, page 104) the differences between the practice exercises and the test exercises became increasingly more potent than the likenesses as the exercises became more complex. Many of the pupils did not react to the identical elements that were present. Explaining in terms of the theories of transfer, we might say that the methods of construction had not been generalized, they had not been freed from the particular figure with which they had been learned. Or we might say that the methods themselves had not been made clear as procedures or patterns of experience. They had been learned as isolated mechanical operations but the essential relationships had not been seen.

When the procedure was analyzed and generalized as shown on page 153, not only was the pattern made clearer and brought to consciousness but it was freed from a particular figure. The transfer was therefore significantly greater in Group D than in Group E which did not have the analysis and generalization (see Table 27, page 154).

Much the same things can be said in connection with the pupils' recognition of the application of terms and theorems in a complex figure. For example, Table 9,

page 112 (showing the results of a test in recognizing alternate interior angles) indicates transfer from a simple figure to complex figures without training. The identical elements were, therefore, potent with some pupils. With others they were not sufficient to cause transfer. Pointing out the zig-zag (page 156 showed the pupils more clearly the essential relation between the lines and the transversal in forming alternate interior angles and gave them a pattern to apply in other figures. Table 30, page 156 shows the improvement after this was done.

Transfer was shown also in connection with the if-then relationship and the meaning of proof. But here we cannot explain with any great satisfaction on the basis of identical elements. The situations are much more complex. It seems more nearly adequate to explain it in terms of meanings, attitudes and ideals; on the basis of bringing the concepts to the level of awareness and generalizing them; or from the point of view of patterns of experience.

It was assumed that the errors made in choosing hypothesis and conclusion from verbal statements were due to the lack of understanding of the if-then relationship. Accordingly, a method was introduced to develop its meaning. Every effort was made to show the pupils of Group D (Group T did not have this training) what is meant by conditions and conclusion in connection with geometric figures. These concepts were associated with the if-part and the then-part of if-then sentences. Pupils were made aware of the fact that they were studying, not the particular sentences under discussion, but the meaning of the if-then relationship in general. By emphasizing the contrast between conditions and conclusion in connection with various geometric figures, the concept stood out from any particular context. It became a pattern of experience in its own right. Consequently, when the pupils of Group D were asked to apply the meaning of the if-then relationship to a new situation—choosing hypothesis and conclusion from a verbal statement, the transfer was greater than in the case of Group T (see Table 31, page 160).

The important point to note in the development of proof described in this study is that the reasoning involved in the geometric proofs was studied as a procedure. "Unless time is set apart for a study of the reasoning processes," says Blackhurst, "no experience with reasoning can be generated. It is not only possible but highly probable that the pupil who has reasoned through all the material of a present day geometry has not in the least added to his experience with reasoning. In other words he has experienced geometry but not reasoning."[12]

In this development, the pupils first learned the meaning of deduction through the medium of syllogisms employing non-geometric material. The process of deduction was analyzed so that the pattern of procedure could be seen. The procedure was then put into words; that is, generalized. When geometric proofs were made, the pupils were shown that these proofs were successive applications of the principles they had learned in connection with syllogisms. They were also shown a method of checking their proofs which again used this procedure. In other words each proof was not an isolated experience but an application of an ever widening pattern.

No attempt has been made in this study to ascertain the effect of improvement in geometry upon the improvement in reasoning outside the field of geometry. The assumption has been that the first step in insuring transfer from geometry to other fields is to improve the reasoning in geometry itself and that, as Mursell says, "when any ability is most intelligently taught and organized for its own sake, it is thereby taught and organized in such a way as will facilitate transfer."[13]

[12] Blackhurst, J. H., "The Educational Values of Logical Geometry," THE MATHEMATICS TEACHER, XXXII (April, 1939), 164.

[13] Mursell, James L., *The Psychology of Secondary School Teaching* (New York: W. W. Norton and Co., 1932), p. 104.

CHAPTER X

SUMMARY

THE purpose of this study was (1) to discover, analyze, and classify errors made by pupils beginning the study of demonstrative geometry and (2) to devise methods of teaching which by making meanings clearer would lessen the number of errors. Consequently the study is in two parts corresponding to the double purpose. The experimentation for Part I was carried through in the fall of 1932 by means of diagnostic tests given over a period of fifty consecutive teaching days to 114 pupils with I.Q.'s ranging from 90 to 146 with a mean I.Q. of 117.9 and standard deviation of 11.7. Almost all of the errors could be classified under three headings, (1) those due to unfamiliarity with figures, (2) those due to not sensing the meaning of the if-then relationship, and (3) those due to meager understanding of the meaning of proof, and the study is restricted to a discussion of these three types of error. The experimentation for Part II was carried on in the fall of 1938 after the methods devised had been tried out on several successive classes. The group used in 1938 consisted of 74 pupils with a distribution of I.Q.'s comparable to that of the original group.

In 1932 the experiment involved five classes and four teachers. Bulletins were issued daily describing in detail the topics and exercises to be covered and the methods to be used. In 1938, three classes and three teachers were involved. In order to keep all factors except method constant in the two experiments, the bulletins given out in 1932 were followed faithfully day by day so far as topics and exercises were concerned. The new factor in the 1938 experiment was the teaching methods which had been developed to deal with the previously mentioned three types of error. The results of certain key tests recorded in Part II showed that these methods caused significant improvement.

The following sections contain a brief summary of the preceding chapters. They consist of the conclusions reached and selected illustrations of the evidence which led to the conclusions.

Complex Figures

This study shows that *even though a pupil may know the meaning of a term, perform a construction, or apply a theorem in connection with a simple figure, he may or may not react correctly to these same things in a complex figure.* After learning how to bisect a horizontal straight line, 6% of the pupils could not bisect the sides of a triangle correctly (page 102). Although all the pupils could bisect a single angle in a given position, 15% could not bisect the angles of a triangle (page 103). They could draw a perpendicular to a line when only a point and a line were involved but as the figure was made more complex the number of pupils making errors reached as high as 52% (page 108). After showing their ability to recognize opposite angles made by two intersecting lines when only the two lines were given, 16% of the pupils did not recognize the opposite angles when the ends of these lines were connected to form two triangles (page 115). They could recognize two sides and the included angle and two angles and the included side in a triangle but as many as 40% made errors when the figure involved overlapping triangles (page 110). When all but 5% of the group were able to identify alternate interior angles in a figure consisting only of two parallel lines and a transversal, a slight complication caused 25% to make errors and further complications increased the number of pupils making errors to as much as 80% (page 112).

The methods devised to minimize the difficulties in connection with figures were based on the assumption that pupils at first see the figures as wholes and that any

procedure which would help them to see the essential parts and dissociate these from the irrelevant would prove effective. Construction problems were analyzed and generalized so that the method would apply to any figure and not merely to a particular figure (page 152). Terms and propositions were illustrated in complex figures as well as in simple figures (page 155) and whenever possible some peculiarity of a configuration like the zig-zags accompanying alternate interior angles was pointed out (page 156). That these methods proved effective is shown by the results recorded in Tables 27, 29, and 30 (pages 154 and 156).

The If-Then Relationship

The if-then relationship is fundamental to postulation thinking. Lack of a clear understanding of its meaning is a real handicap to students beginning demonstrative geometry. *Many students do not understand the logical implication of the if-then relationship before it is developed in the geometry class, and the growth of the concept is slow after development is begun.*

Out of 74 pupils in the 1938 group, 61 said that there was no difference between the converse statements. *If two sides of a triangle are equal, the angles opposite those sides are equal,* and *If two angles of a triangle are equal, the sides opposite those angles are equal.* And 55 of them said that those two statements together meant the same thing as *In an isosceles triangle two sides and the two angles opposite them are equal* (page 157). When asked to construct figures to test the truth of the two converse statements just stated only 58% of the 1932 group sensed the meaning of the if-then relationship well enough to make two sides equal for the first statement and two angles equal for the second (page 119). Difficulties of this kind persisted in the case of fully one tenth of the group even after six days' practice (page 121). A multiplicity of errors was made by 59% of the pupils in drawing a figure and writing the hypothesis and conclusion in terms of that figure for a rather simple verbal statement (page 122).

The meaning of the if-then relationship was developed concretely with the 1938 group by means of constructions. Pupils saw that when they did certain things in making a figure certain other things resulted. They learned to feel the difference in category between the relationships they put into a figure—the things over which they had control—and the relationships which resulted without any action on their part. Finally the difference in these two categories was associated with the difference between given conditions and conclusion, between the if-part and the then-part of a sentence (page 158 ff). Because of this development the pupils in 1938 showed significant superiority over the 1932 group in their ability to write the hypothesis and conclusion in terms of a figure when given a proposition (page 160).

Meaning of Proof

The meaning of proof was discussed under four headings, Deduction, Meaning of Hypothesis, Acceptable Reasons, and Proofs of Exercises. *It was found that a large percentage of pupils, when they begin the study of demonstrative geometry, have little conception of what is involved in making a formal deduction.* An analysis of a test on syllogisms resulted in the following conclusions (page 126).

(1) Although pupils can make a deduction when simple everyday situations without any complications are given, they cannot necessarily do so when there are complications even in such simple situations.

(2) Pupils need to learn that a conclusion can be drawn only when all the conditions are fulfilled.

(3) They must learn to analyze a statement to find out what the conditions are that must be fulfilled, and they must be able to see whether all the conditions have been fulfilled.

(4) They must learn that when the conditions are fulfilled the only conclusion

that can be drawn is the one stated in the general statement (first premise). Pupils certainly are not ready to make demonstrations in geometry until they have mastered these things.

Pupils do not readily grasp the concept of holding to the data. Without careful training they will give more weight to the appearance of a figure than to the hypothesis. In a multiple choice test (Test 21, page 128) in which only one of five choices followed from the data but in which all five choices appeared true in the figure, a large number of pupils made errors (36% on one exercise). When this test was given, the pupils had been working for sixteen days, with exercises requiring them to hold to given conditions.

Pupils are handicapped in their geometric reasoning by most of their previous experience in intuitive thinking concerning things in general. It has been their experience to make deductions using any part of the total situation that appeals to them. In geometry they are definitely restricted in their reasoning to the data, definitions, and propositions assumed or previously proved. *In order to understand the significance of a proof in geometry pupils must learn at the beginning of their formal work that they are restricted as to the reasons they may use.* Left to their own devices pupils will give reasons made up to suit the occasion (page 130).

In the final test on original exercises (Test 25, page 133), there were only a few errors which could not be classified under the three headings discussed here. No conclusion could be drawn as to the type which caused the most errors, the kind of error depending upon the exercise (see distribution of errors, page 134).

The 1938 group of pupils was given a careful development of the meaning of proof. Deduction was introduced by means of syllogisms the correctness of which could first be checked by studying the concrete situation. The syllogisms were then analyzed to discover means by which the statements themselves gave the cue to their rightness or wrongness. The result of this discussion was the following summary concerning deductions (page 163).

1. A deduction consists of three statements, (a) a general statement, (b) a specific statement which is an application of the general statement, and (c) a conclusion.

2. No conclusion can be drawn from the first two statements unless the second statement fulfills exactly the conditions of the first statement: that is, *fits* exactly the if-part of the first statement.

3. If the second statement fulfills the conditions of the first statement, then there always is a conclusion and this conclusion must follow exactly the then-part of the first statement.

Postulates were introduced inductively and listed (page 165 ff). Theorems were listed as they were proved. Pupils were led to see that the purpose of a proof was to see how new theorems grew out of the postulates and theorems. If, therefore, they used reasons not on their lists they were going contrary to the purpose of a proof.

The necessity of holding definitely to the data was shown by drawing special and general figures with the same data (page 129). Pupils could see that many different shaped figures could be drawn with the same hypothesis and that the only things true of all of them were those resulting from what was specifically given.

When the first formal proofs were introduced they were shown to be successive applications of the principles learned in connection with syllogisms (page 168 ff). Since pupils had learned a method of checking the correctness of a syllogism, they now had a method of checking a geometric proof. They were urged to check each proof by asking themselves the following three questions (page 169).

1. Are all the reasons acceptable ones?
2. Are all the conditions of each reason fulfilled?
3. Is the conclusion of each reason followed exactly?

If the answers to all these questions was in the affirmative the pupil could be reasonably sure that his proof was correct. If at least one answer was negative, something was wrong and further thought required.

The emphasis throughout was on the method of procedure and only secondarily on the geometry. To be sure, the geometry had to be learned and organized because it was the basis of the reasoning. But the aim was never to allow the details of geometry to obscure the analysis of the thought processes.

Because of this careful discussion of the meaning of proof, the 1938 group showed significant superiority over the 1932 group in proving original exercises (see Table 34, page 170).

In Chapter IX, the methods introduced in this study were shown to be in accordance with the laws of the transfer of training. These methods should be, therefore, not only the foundation for better results in geometry itself but helpful in building up a course of study which will allow for the carryover of the fundamental procedures to other fields.

Summer Meeting of The National Council!

THE SUMMER meeting of the Council will be held at Milwaukee from July 1 to 3 inclusive. The theme is "How Mathematics Serves Our Community." See the May issue of the TEACHER for the complete program.

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◆ THE ART OF TEACHING ◆

A Simple Blackboard Ellipsograph

By HAYM KRUGLAK
Milwaukee Vocational School, Milwaukee, Wisconsin

THE SIMPLEST and best known method of drawing an ellipse is based on the fundamental theorem in conics that in any ellipse the sum of the focal radii is constant and equal to the major axis of the ellipse. This method is widely used by gardeners in laying out elliptical beds, by pattern-makers, metal workers, artists, and others.

In practice two pins, nails, or thumbtacks are driven in at F and F', the foci of the desired ellipse, (Fig. 1). A looped string of length $2a+2c$ is made to enclose F and F'. A moving marker P pressed tightly against the string traces out an ellipse.

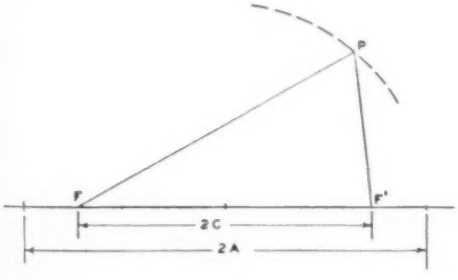

FIG. 1

However, when it is desired to draw an ellipse on a hard surface, such as a blackboard, there arises the difficulty of establishing F and F' without mutilating the surface.

The writer found that inexpensive rubber suction cups adhere well to a blackboard. A screw and nut arrangement changes the convexity of the rubber, thus facilitating attachment and removal. These cups may be bought in any auto supply store or 5-and-10 cent store.

Two such cups are attached at F and F' respectively. A looped piece of string is allowed to slide about F and F'. A fairly accurate ellipse can be drawn with one continuous sweep of a piece of chalk at P.

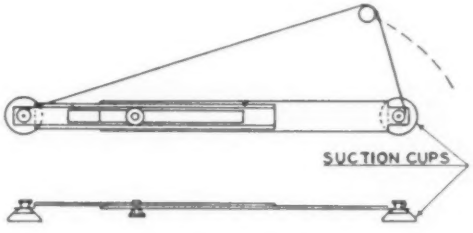

FIG. 2. Blackboard Ellipsograph

Figure 2 shows a simple ellipsograph in which the cups are mounted on a sheet metal frame. The distance between the cups, and hence the eccentricity for a given piece of string, can be changed by means of a simple sliding arrangement.

By using a single suction cup and a looped string very excellent circles can be drawn. The instructor can supply his class with inexpensive compasses at a low cost.

WHEN you can measure what you are speaking about and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind.—LORD KELVIN.

EDITORIALS

In the March issue, editorials appeared concerning the Past President of the National Council of Teachers of Mathematics, Professor H. C. Christofferson, and the new President, Miss Mary A. Potter. It is fitting that the readers of The Mathematics Teacher should hear from each of these industrious Council workers. The two editorials below have been prepared especially for this issue by them. They will be of interest to all of the membership of the National Council and to readers of The Mathematics Teacher.

W. D. R.

Looking Forward

At twenty years of age, with a childhood and adolescence which shows great promise by virtue of its high A.Q. (achievement quotient), The National Council of Teachers of Mathematics faces an era of challenging possibilities for widespread influence and vital service. The Council is dedicated to the task of helping to provide for children the best possible education. In a civilization that is daily becoming increasingly scientific and mathematical, as well as social, it would indeed be short-sighted folly to allow children to leave our schools without a mathematical education which will help them to analyze quantitative situations accurately; to perform necessary computations without error; to appreciate and to understand graphs, tables, formulas, and other mathematical concepts commonly used to present ideas; and to think clearly and with mathematical precision in quantitative and other situations.

It is unfortunately true that many mathematics teachers are in a rut, college and secondary even more so than elementary teachers. We have held too much as sacred the traditional organization and content, failing to utilize adequately the abundant possibilities inherent in mathematics for enriching the lives and awakening the minds of children. As a consequence educational leaders have felt that the education contributions of mathematics have been grossly over-rated. Many children are now leaving our schools with a meager and entirely inadequate command of mathematical concepts, skills and understandings.

It is therefore fitting that we highly resolve as a National Council of Teachers of Mathematics, beginning its twenty-first year of existence, to go forth with renewed vigor to help all mathematics teachers to give all children the most useful and helpful mathematical education possible. This goal can never be attained without alert and loyal cooperation by all who are interested in educating children. "Every mathematics teacher in the United States a member of the Council and actively participating in its program" should be our slogan if we really wish to insure to children adequate preparation to face and to solve the problems of a complex civilization.

As the retiring president it would be fitting for me to pay tribute to those who have helped so loyally in the past in carrying on the activities of the Council. I know however, that they would rather have me utilize this space, not to look toward our past, but to use the past so as the better to envisage the future. The joy of working on so vital and challenging a problem is the chief and most satisfying reward for these pioneer men and women. Their names are well known and their work is indelibly recorded in the annals of the Council and in the hearts of those who teach children mathematics. May we as mathematics teachers follow their cour-

ageous lead, may we share with them the joy of continued pioneering in mathematical education for children, and may we reap with them the reward of satisfaction for helping to give children a complete and functional education.

H. C. CHRISTOFFERSON.

What of the Future?

SEPARATED as we are and even without benefit of radio, may we gather together for a "fireside chat" about the affairs of the Council.

To many of us the Council has been the chief souce of inspiration for the interpreting of mathematics to our pupils. We owe a debt which cannot be repaid to the far-seeing men and women who have made possible the Council as it exists today. From a modest beginning but with the selfless zeal of pioneers they have edited THE MATHEMATICS TEACHER and the *Yearbooks*, written articles, organized meetings, prepared and delivered addresses, established branches, worked on various committees, created and administered finances, have done the thousand and one tasks both great and small that are necessary to make an organization live and grow.

The conventions have not only kept us in touch with the more advanced thinking and practice in the teaching of mathematics but have afforded us an opportunity to discuss our own problems more intimately with other teachers facing similar situations. We prize the friendliness as well as the information on teaching trends that have characterized meetings of the Council, making a powerful stimulant for better teaching.

Articles in the TEACHER and *Yearbooks* have been of sufficient importance to form a sizeable fraction of the literature on the teaching of mathematics available for courses in teacher training and for the education of teachers in service. Such is the excellent achievement made in the last twenty years. It is for us to face present conditions and to carry on.

The chaos of the troubled world of 1940 and the perplexing economic conditions changing rapidly even before they are understood and before humanity can make any reasonable adjustment, have a profound influence upon education. Among the but partially solved problems challenging the teacher of mathematics the following may be listed: the distrust by educators of the importance of mathematics in a world increasingly dependent upon mathematics; the problem of the most efficient mathematical education to prepare pupils for successful participation in this new mathematically-controlled scientific world; the problem presented by the huge influx into the secondary school of young people formerly absorbed by industry; many problems created by the greater spread of accomplishment in a single grade due to the present practice of promotion by age rather than by attainment; the heavier teaching loads necessary because of decreased school budgets and increased school population.

The degree to which these problems will be solved depends upon the amount of united effort put forth by all of the Council members.

In order that more people may benefit from the inspiration of meeting together, the Council through its Board of Directors and program committee continues to offer you three conventions in various areas during the year: one in Milwaukee when the summer N.E.A. convenes; a meeting in Baton Rouge during the Christmas vacation, which is but the second time we have met in the South; and the annual February meeting (place yet to be decided).

In addition to the ever-helpful TEACHER, the editorial committee promises a program of unusual merit. The Report of the Joint Commission, which every teacher of mathematics will wish to read with critical care, will be followed next year by the im-

portant Report of the Arithmetic Committee. Also, a second pamphlet written in popular style and similar to *Numbers and Numerals* will soon appear, a gift to members of the Council and priced low enough to be available to everyone.

The work done by those directors in charge of the state representatives, affiliated clubs, and speakers bureau has been so efficient that it has outgrown the reasonable time limits possible for people with a full-time teaching job, and has been delegated to the recently employed executive secretary. The various excellent committees are actively functioning; reports of their finding will appear at intervals in the TEACHER.

Perhaps you wonder what you as a member can contribute. The possibilities of service that you can give are very great. In addition to the usual attendance of meetings, working on committees, appearing on various programs, writing articles, cooperating with the still young speakers bureau, you can send in your criticisms and suggestions. This is your organization which we hope is of practical benefit to each member; the picture presented by your particular needs and problems will give a sound plan for extending its usefulness.

May we, the National Council of Teachers of Mathematics, work together to enrich the lives of the youth of America through their better understanding of its mathematical aspects. MARY A. POTTER.

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IN OTHER PERIODICALS

By NATHAN LAZAR
The Bronx High School of Science, New York City

1. Addleston, Lorraine W., "The Mathematics Teacher Faces a New Junior High School." *High Points*, Vol. 21, No. 10, December, 1939, pp. 67–70.

The teacher of mathematics in the seventh grade is often faced with the realization that the students are not adequately equipped with certain indispensable mathematics techniques and facts. The two alternative solutions open to him are (a) to begin from the beginning and re-teach the fundamentals to every one; or (b) to enter upon a program of remedial teaching.

The disadvantages of the former plan are pointed out. The latter plan is recommended and detailed directions are given for carrying it out successfully. The desirable by-products of the latter plan are also indicated.

2. Brown, Claude H., "A Curriculum Unit on Non-Euclidean Geometry." *School Science and Mathematics*, 40: 128–136. February, 1940.

"The aim of this unit on non-Euclidean geometry is twofold. On the one hand, it is intended to emphasize in as striking a manner as possible, that the conclusions of any logical argument depend upon the assumptions which form the basis for that argument. On the other hand, it is hoped that it will suggest to the student new areas of interest to explore."

By carefully chosen questions, the writer leads the student to a realization of the difficulties inherent in the concept of parallel lines. A brief historical description and a short bibliography are also included.

3. Evans, John Ellis, "Why Logarithms to the Base *e* Can Justly Be Called Natural Logarithms." *National Mathematics Magazine*, 14: 91–95. November, 1939.

The following three reasons are usually given for calling logarithms to the base *e* natural logarithms: (a) *e* is a quantity which arises frequently and unavoidably in nature; (b) natural logarithms have the simplest derivatives of all the systems of logarithms; (c) in the calculation of logarithms to any base, logarithms to the base *e* are first calculated, then multiplied by a constant which depends on the system being calculated.

Each of the reasons is discussed at some length, and appropriate quotations and bibliographies are supplied.

4. Ford, L. R., "An Alignment Chart for the Quadratic Equation." *American Mathematical Monthly*, 46: 508–511. October, 1939.

A description of the theory underlying the construction and use of a nomographic chart for solving quadratic equations of the type $x^2 = ax + b$. The chart itself is also reproduced.

5. Fraenkel, Abraham Adolf, "Natural Numbers as Cardinals." *Scripta Mathematica*, 6: 69–79. June, 1939.

The various theories dealing with the place of positive integers in the structure of mathematics are briefly mentioned. This is followed by a clear exposition of the natural numbers from the points of view of (a) the positional notation; (b) the concept of cardinal number; (c) the ordering of numbers according to magnitude. A short bibliography is also given.

6. Keyser, Cassius Jackson, "The Role of Mathematics in the Tragedy of Our Modern Culture." *Scripta Mathematica*, 6: 81–87. June, 1939.

The literatures of mathematics and of mathematical physics have been growing so rapidly that even the highly intellectual people of the world are, with few exceptions, quite unable to read them. This tragedy of our modern culture, though it cannot be ended, can be mitigated by the writing of intelligent, though popular, expositions of these subjects. A number of such books are mentioned. It is recommended that "a convention or a congress held for the express purpose of exploring such means could hardly fail to be of immense cultural service to the great 'nation' of intellectual non-mathematicians."

7. Sister M. Thomas à Kempis, "An Appreciation of Sophie Germain," *National Mathematics Magazine*, 14: 81–90. November, 1939.

A brief description of the life and work of a

famous woman mathematician who lived during the years 1776–1831.

The author rightly complains that few of the histories of mathematics have given Sophie Germain her due. She seems, however, to have overlooked a highly appreciative tribute accorded her by E. T. Bell in his "Men of Mathematics," pages 261–263.

8. Rex, Earl C., "Construction of a Demonstration Slide Rule." *School Science and Mathematics*, 40: 161–164. February, 1940.

A complete description of the materials, technique, etc., necessary for the construction of a demonstration slide rule. A diagram and photograph are also included.

9. Seidlin, Joseph, "More About Better Mathematics." *National Mathematics Magazine*, 14: 96–99. November, 1939.

Some very penetrating observations on the varieties of philosophies and objectives of the teaching of mathematics. The ideals of the right wingers, the left wingers, and of the centrists are summarized in pithy phrases.

The article ends on a melancholy note. The centrists "are keenly, albeit unhappily, aware of the fact that the greater part of human ills and suffering and discomforts is not caused by lack of understanding, reasoning, analysis, or scientific detachment and, therefore, cannot be reduced or eliminated even if it were possible to make a high grade mathematician out of every man, woman, and child in our society. In other words, social progress depends on many, many factors that even an ideal study of an ideal subject matter in mathematics does not involve."

10. Seidlin, Joseph, "What Price Isolation?" *National Mathematics Magazine*, 14: 66. November, 1939.

An urgent appeal to mathematicians not to remain smugly aloof from the educational turmoil of today.

11. Waits, B. Lewis, "The Quadratic Formula." *School Science and Mathematics*, 40: 145. February, 1940.

The familiar quadratic formula can also be derived by substituting the complex number x_1+iy_1 for x in the general quadratic equation

$$ax^2+bx+c=0.$$

This method is recommended to supplement, of course, the usual one; and, according to the author, has been found very successful with the advanced students in second year algebra. It will undoubtedly make an interesting program for a mathematics club.

THE average boy or girl is afraid of mathematics. To them the study of mathematics is the traditional bugaboo. They are afraid of it just as ignorant people are afraid of foreigners or of people whom they do not understand. Haven't you feared the deep water before you learned to swim? On the other hand, I have known a great many high school students who preferred mathematics to anything else on their programs, pupils whose sole interest in school was mathematics. They made a friend of this subject, as you do of some people you meet. Or else, they took mathematics as a game to be played. They learned the rules of the game and played it, played it hard. You should not be afraid of mathematics. Take an interest in it. You will learn to love it. In future years, it will reciprocate your love financially and spiritually.—MURRAY J. LEVENTHAL.

Lazar's Book Sold Out

DR. NATHAN LAZAR'S book *Logic in Geometry* is no longer available from the publisher, THE MATHEMATICS TEACHER, or from the Bureau of Publications, Teachers College, Columbia University. The original printing is sold out, and no further one is planned.

NEWS NOTES

Professor Julian L. Coolidge, senior member of the Department of Mathematics at Harvard University, will retire on September 1, 1940, with the titles Professor of Mathematics Emeritus and Master of Lowell House Emeritus. Professor Coolidge has been a member of the Department of Mathematics for forty years and Master of Lowell House for ten.

There will be a meeting of the Mathematics Section of the Private Schools Association of the Central States at the Milwaukee Country Day School on Saturday, May 4, at 10:00 A.M., Central Standard time.

Program details to be announced later.

Complimentary luncheon at 12:00.

Afternoon discussion begins at 1:30.

PAUL W. WATERMAN, Chairman

The March meeting of the Women's Mathematics Club of Chicago and Vicinity was held at the Normandy House Saturday, the 16th. Luncheon was served at twelve o'clock.

Miss Lillian Arends, of Proviso Township High School, gave a short talk on "Enrichment Material."

One of our own members, Miss Virginia Modesitt, of Wright Junior College, was the speaker of the afternoon. She chose "Euclid" as her topic.

DOROTHY B. LANDERS

The Association of Mathematics Teachers of New Jersey held a joint conference of elementary, junior high, and senior high school teachers of mathematics on March 9, at the State Teachers College, Montclair.

An unusually thorough demonstration of films and slides related to the teaching of mathematics was conducted throughout the day. Other exhibits included mathematical equipment, models, and multi-sensory aids.

Morning sessions were devoted to demonstration classes in the teaching of geometry. Leaders in charge of these sessions were:

1. Fourth Grade Level: Miss Elizabeth Shaner, Columbia School, East Orange, N. J. *Presiding Officer*, Dr. Foster E. Grossnickle.

2. Junior High School Level: Miss Rachael Peters, Teaneck High School, Teaneck, N. J. *Presiding Officer*, Miss Mary C. Rogers.

3. Informal Plane Geometry Tenth Grade Level: Mr. George N. Ramage, Linden High School, Linden, N. J. *Presiding Officer*, Ferdinand Kertes.

4. Formal Demonstrative Plane Geometry Tenth Grade Level: Dr. Virgil S. Mallory, Professor of Mathematics, State Teachers College, Montclair, N. J. *Presiding Officer*, Ferdinand Kertes.

5. Sixth Grade Level: Mrs. Madeline Dempsey Gardner, Forest Avenue School, Glen Ridge, N. J. *Presiding Officer*, Dr. Foster E. Grossnickle.

Dr. Robert Lee Morton, of Ohio University, made the principal address of the conference in the afternoon following a discussion luncheon.

The discussion luncheon topics all related to the general theme "What Can Our Association Do to Render Greater Service to the Teachers of Mathematics?" The table topics and leaders were as follows:

SECONDARY LEVEL

1. Collection of Life Problem Materials of Current Interest. Mr. Paul C. Clifford, State Teachers College, Montclair, N. J.

2. Social Mathematics on the 12th Year Level. Mr. Howard F. Hart, Head of Department of Mathematics, East Orange High School, East Orange, N. J.

3. Collection of Problem Material from the Social Studies. (2 groups.) Mr. Francis P. Rice, Mount Hebron Junior High School, Montclair, N. J., and Mr. Hubert B. Risinger, Vernon L. Davey Junior High School, East Orange, N. J.

4. Teaching Algebra for Meaning. Mr. Joseph G. Shuttlesworth, Summit Senior High School, Summit, N. J.

5. Development of Guidance Attitude in the Mathematics Classroom. Mr. Harold A. Zantow, Stevens Hoboken Academy, Hoboken, N. J.

6. Club Programs, Mathematics Club Bulletin and Procedures. Mr. Jerome Schlosser, McKinley Junior High School, Newark, N. J.

7. Modes of Thinking in Mathematics. Dr. David R. Davis, State Teachers College, Montclair, N. J.

8. Teaching for Transfer. Dr. Virgil S. Mallory, Professor of Mathematics, State Teachers College, Montclair, N. J.

9. General Mathematics. Mr. George N. Ramage, Linden High School, Linden, N. J.

10. The Colliton Scoring System in Football. Mr. J. W. Colliton, Head of Department of

Mathematics, Trenton Senior High School, Trenton, N. J.

11. Possibilities and Limitations of Multi-Sensory Aids as a Means of Improving Instruction in Mathematics. Mr. Joseph Hilsenrath, Townsend Harris High School, New York City.

12. The High School Teachers Interest in a Background of Higher Mathematics. Professor Albert E. Meder, Jr., New Jersey College for Women, New Brunswick, N. J.

13. Importance of Mathematics in Preparation for a Career in Science or Engineering. Professor Charles O. Gunther, Stevens Institute of Technology, Hoboken, N. J.

14. Revision of Mathematics Courses of Study. Miss Mary C. Rogers, Roosevelt Junior High School, Westfield, N. J.

15. Revision of Mathematics Courses of Study. (Senior High School Level.) Mr. Chauncey W. Oakley, Manasquan High School, Manasquan, N. J.

16. Selling Mathematics. Mr. Howard F. Fehr, State Teachers College, Montclair, N. J.

17. Common Mathematical Fallacies in the Use of Graphs. Mr. Michael R. McGreal, Principal Barringer High School, Newark, N. J.

Elementary Level

18. Promoting the Study of Local Problems in the Teaching of Mathematics. Mr. Paul Poehler, Principal Columbian School, East Orange, N. J.

19. Social Values of Mathematics. (2 groups.) Mr. Gordon Matthews, State Teachers College, Paterson, N. J., and Mrs. Viola R. Wriggins, Walnut Street School, Woodbury, N. J.

20. Possibilities and Limitations of Multi-Sensory Aids as a Means of Improving Instruction in Mathematics. Miss Frieda Harrell, Vernon L. Davey Junior High School, East Orange, N. J.

21. Teaching Arithmetic for Meaning. Mr. John W. Dickey, State Teachers College, Newark, N. J.

22. Collection of Life Problem Materials. Miss Anna Graffan, Principal Lincoln School, Lyndhurst, N. J.

23. Revision of Mathematics Courses of Study. Mrs. Jane Plenty, State Teachers College, Newark, N. J.

Combined Elementary and Secondary Levels

24. Prognostic, Diagnostic, Progress and Achievement Tests in Mathematics. Dr. Martha Downs, State Teachers College, Newark N. J.

25. Use of Materials and Devices as Motivating Aids in the Teaching of Mathematics. Dr. E. H. C. Hildebrandt, Assistant Professor of Mathematics, State Teachers College, Montclair, N. J.

The following is the program of the Mathematics Section for the Colorado Educational Association Convention held in Denver, Colorado, on October 26, 1939:

Table discussion during luncheon meeting:

Topic: "Why Teach Mathematics?"

Table 1—From the point of view of junior high, Wendell Wolf, discussion leader.

Table 2—From the point of view of junior high, Margaret Aylard, discussion leader.

Table 3—From the point of view of senior high, Farnum St. John, discussion leader.

Table 4—From the point of view of senior high, Florence Barnard, discussion leader.

Table 5—From the point of view of college, Claribel Kendall, discussion leader.

Table 6—From the point of view of college, J. C. Fitterer, discussion leader.

Topic: "What Mathematics Shall We Teach?"

Table 7—From the point of view of junior high, Sally Reeves, discussion leader.

Table 8—From the point of view of junior high, Ray T. Ashbaugh, discussion leader.

Table 9—From the point of view of senior high, Lillian Sullivan, discussion leader.

Table 10—From the point of view of senior high, Courtland Washburn, discussion leader.

Table 11—From the point of view of college, Dwight F. Gunder, discussion leader.

Table 12—From the point of view of college, W. V. Lovett, discussion leader.

Address: "Mechanics of Engineering," by Dr. Fred Carstarphen, President of the Colorado Society of Engineers.

The Mathematics Section of the Bay Section of the California Teachers Association will meet on April 20, at International House, in

Berkeley. Dr. Eric Temple Bell of the California Institute of Technology will speak.

The Men's Mathematics Club of Chicago and the Metropolitan Area met March 15 for dinner and an evening of listening to addresses and discussion.

G. E. Hawkins, president of the club, gave a report of the National Council of Teachers of Mathematics Convention at St. Louis, February 22 and 23.

Dr. M. L. Hartung spoke on "Isosceles Triangles and Their Offspring." Dr. Hartung has recently been made Assistant Professor of Mathematics at the University of Chicago. He was formerly on the staffs of Ohio State University and the University of Wisconsin High School.

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